18.155 LECTURE 5
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Abstract. Notes before and then, eventually, after lecture.

Read: Chapter 3, section 8 – but it isn't written. I will try to write something up before (but more likely after) the lecture. See below.

Before lecture

- Duality and $(H^s(\mathbb{R}^n))' = H^{-s}(\mathbb{R}^n)$.
- Distributions independent of one variable.
  If $\psi \in \mathcal{S}(\mathbb{R})$ then $\int_{-\infty}^{x} \psi(t)dt \in \mathcal{S}(\mathbb{R})$ iff $\int \psi = 0$.
  If $\phi \in \mathcal{S}(\mathbb{R})$ then $\phi(x) = \frac{1}{\sqrt{2\pi}}(\int \phi)\exp(-\frac{x^2}{2}) + \psi$, $\psi \in \mathcal{S}$, $\int \psi = 0$.
- Can we solve $\frac{du}{dx} = f$ in tempered distributions?
  What we need is $u(-\frac{d\phi}{dx}) = f(\phi)$.
- Holomorphic functions of one variable. Uniqueness of holomorphic continuation. Holomorphic functions valued in $\mathcal{S}'(\mathbb{R})$, weak holomorphy.
- The tempered distributions, $x_+^z \in \mathcal{S}'(\mathbb{R})$ $\operatorname{Re} z > 0$.
- Holomorphy of the integral $\int_0^\infty x^z \phi(x)$, $\phi \in \mathcal{S}(\mathbb{R})$.
- The identity $x\partial_x x_+^z = zx_+^z$, $\operatorname{Re} z > 1$.
- $\int_0^\infty x^z\phi(x) = \frac{-1}{z+1}\int_0^\infty x^{z+1}\frac{d}{dx}\phi(x)$, $\operatorname{Re} z > 1$.
- Holomorphic extension of x_+^z to $z \in \mathbb{C} \setminus (-\mathbb{N})$.
- Delta distributions at the origin.
- Residue at $z = -k$.
- Regularized value at $z = -k$.
- $x_-^z = (-x)_+^z$.
- $(x + i0)^z = \exp(z\log(x + i0)) = x_+^z + e^{i\pi z}x_-^z$ is entire.
- Linear transformations and $\mathcal{S}'(\mathbb{R}^n)$.
- All homogeneous tempered distributions on $\mathbb{R}$ – we need some more theorems.
- Homogeneity in higher dimensions, similarly but more complicated. We will do it.

After lecture

I did not discuss the solvability of d/dx – I will do this first on Tuesday and show that

$$\begin{gathered}\frac{d}{dx} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) \text{ is injective with range of codimension 1}\\ \frac{d}{dx} : \mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R}) \text{ is surjective with null space of codimension 1.}\end{gathered} \tag{1}$$

Here is a preliminary version of the treatment of homogeneous distributions in dimension one which will make it to the notes sometime soon; I may need to squash some typos first.

On $\mathbb{R}$ for each $z \in \mathbb{R}$, $\operatorname{Re} z > 0$ the functions (using the standard branch of the logarithm)

$$x_+^z = \begin{cases} x^z = \exp(z \log x) & x > 0 \\ 0 & x \le 0 \end{cases} \text{ and } x_1^z = \begin{cases} 0 & x \ge 0 \\ (-x)^z = \exp(z \log(-x)) & x < 0 \end{cases}$$

are continuous and of slow growth, $|x_\pm^z| \le |x|^{\operatorname{Re} z}$, so they define tempered distributions as usual,

$$x_\pm^z(\phi) = \int_0^\infty x^z \phi(\pm x) dx.$$

Definition 1. A map $u_z : \Omega \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ on an open set $\Omega \subset \mathbb{C}$ is said to be (weakly) holomorphic if for each $\phi \in \mathcal{S}'(\mathbb{R}^n)$ the composite maps

$$\Omega \ni z \longmapsto u_z(\phi)\mathbb{C},\ \phi \in \mathcal{S}(\mathbb{R}^n), \tag{2}$$

given by evaluating on any test function, are holomorphic in the usual sense.

Lemma 1. *The functions $x_\pm^z$ are holomorphic with values in $\mathcal{S}'(\mathbb{R})$ in the half-plane $\Omega = \{z; \operatorname{Re} z > 0\}$.*

Proof. This is just saying that for each $\phi \in \mathcal{S}(\mathbb{R})$ the integrals

$$\int_0^\infty x^z \phi(\pm x) dx,\ \phi \in \mathcal{S}(\mathbb{R}) \tag{3}$$

are holomorphic as functions of z. In fact this follows from the holomorphy of x^z in the usual sense. The derivative of x^z in $x > 0$ with respect to $\operatorname{Re} z$ is $x^z \log x$ and similarly for $\operatorname{Im} z$. For $\operatorname{Re} z > 0$ the difference quotients involved converge uniformly on $[0, \infty)$ when multiplied by ϕ so in fact the integrand, as a function of $\operatorname{Re} z + i \operatorname{Im} z$ is continuously differentiable and satisfies the Cauchy-Riemann equations

$$(\partial_{\operatorname{Re} z} - i\partial_{\operatorname{Im} z}) \int_0^\infty x^z \phi(\pm x) dx = 0. \tag{4}$$

Thus the family of distributions is (weakly) holomorphic. □

One reason we are interested in the holomorphy of this family is the 'uniqueness of holomorphic continuation'. If $\Omega_2 \supset \Omega_1$ are open subsets of $\mathbb{C}$ with Ω_2 connected and u is holomorphic on Ω_1 then there can be at most one holomorphic function v on Ω_2 equal to u on Ω_1. Of course in general there is no such function but the point is that two holomorphic functions on the same connected open set which are equal on an open neighourhood of any one point are equal throughout the open set.

To see why this is of interest, suppose for the moment that $\operatorname{Re} z > 1$. Then as a function x_+^z is continuously differentiable (in x for each fixed z) and satisfies

$$\frac{dx_+^z}{dx} = z x_+^{z-1}. \tag{5}$$

Changing the variable to $z + 1$ this can be written

$$x_+^z = \frac{1}{z+1} \frac{d}{dx} x_+^{z+1},\ \operatorname{Re} z > 0. \tag{6}$$

This same equation holds in the distributional sense, so for any $\phi \in \mathcal{S}(\mathbb{R})$,

$$x_+^z(\phi) = \frac{1}{z+1} x_+^{z+1}(-\frac{d\phi}{dx}), \operatorname{Re} z > 0. \tag{7}$$

This of course amounts to an equality of integrals as in (3). The relevant observation is that the term on the right is actually holomorphic in $\operatorname{Re}(z+1) > 0$, i.e. $\operatorname{Re} z > -1$. Since it is equal to the holomorphic function on the left for $\operatorname{Re} z > 0$, that function must itself be holomorphic in $\operatorname{Re} z > -1$. This argument can be iterated.

Lemma 2. *The pairing $x_\pm^z(\phi)$ for any $\phi \in \mathcal{S}(\mathbb{R})$, extends to be holomorphic in $\mathbb{C} \setminus -\mathbb{N}$ with only simple poles at the negative integers and defines a tempered distribution for each $z \notin -\mathbb{N}$.*

Proof. We have just seen that the left side of (7) is holomorphic in $\operatorname{Re} z > -1$ for each ϕ, so the right side is holomorphic in $\operatorname{Re} z > -1$ except for a (possible) pole at $z = -1$. This argument can be continued.

Alternatively, we can iterate the formula (7) itself, initially in $\operatorname{Re} z >> 0$, to see that for any $k \in \mathbb{N}$,

$$x_+^z(\phi) = \frac{1}{(z+1)\dots(z+k)} x_+^{z+k}((-1)^k \frac{d^k\phi}{dx^k}). \tag{8}$$

This shows the existence of the meromorphic extension to $\operatorname{Re} z > -k$ directly and it also follows that for any $z \notin -\mathbb{N}$, with $\operatorname{Re} x > -k$, there is a bound

$$\begin{aligned} |x_+^z(\phi)| &= |\frac{1}{(z+1)\dots(z+k)} x_+^{z+k}((-1)^k \frac{d^k\phi}{dx^k})| \\ &\le C_z \sup_{x \ge 0} \|(1+|x|^2)\frac{d^k\phi}{dx^k}| \le C_z \|\phi\|_{(k+2)}, \end{aligned} \tag{9}$$

so x_+^z is indeed a tempered distribution.

Of course similar arguments apply to x_-^z. ☐

As a function for $\operatorname{Re} z > 0$, x_+^z is positively homogeneous of complex degree z. This just means that if $a > 0$ then

$$(ax)_+^z = a^z x_+^z.$$

Thinking of $x_+^z = \mu_z$ as a distribution we can see this in a weak form as

$$\begin{aligned} \mu_z(x)(\varphi(\frac{x}{a})) &= \int_0^\infty x^z \varphi(\frac{x}{a}) dx \\ &= a^{z+1} \int_0^\infty t^z \varphi(t) dt \\ &= a^{z+1} \mu_z(\phi) \end{aligned} \tag{10}$$

Thus we *define* homogeneity of degree z for a tempered distribution in one dimension by requiring that the identity

$$u(\phi(\frac{\cdot}{a})) = a^{z+1} u(\phi), \ \forall\ a > 0,\ \phi \in \mathcal{S}(\mathbb{R}) \tag{11}$$

hold.

Proposition 1. *For any $z \in \mathbb{C}$ there is precisely a 2-dimensional space of homogeneous (tempered) distributions on $\mathbb{R}$ of degree z; for $z \notin -\mathbb{N}$ it is spanned by x_+^z and x_-^z*

Proof. Since the identity (10) holds for $\operatorname{Re} z > 0$ and both sides are holomorphic in z it too must hold for all $z \notin -\mathbb{N}$. So to see that the space of homogeneous distributions is at least 2-dimensional for each such z we only need to show that x_+^z and x_-^z are linearly independent.

To see this, observe that if $\phi \in \mathcal{S}(\mathbb{R})$ vanishes in say $|x| < 1$ then the integrals in (3) are both defined for all $z \in \mathbb{C}$ and so define entire, i.e. everywhere holomorphic, functions which must reduce to $x_\pm^z(\phi)$ by the uniqueness of analytic continuation. Since $x_+^z(\phi) = 0$ if $\phi(x) = 0$ in $x > -1$ and $x_-^z(\phi) = 0$ if $\phi(x) = 0$ in $x < 1$, it follows that these two functionals are linearly independent for any $z \notin -\mathbb{N}$.

Thus the space of homogeneous distributions of degree z must be at least 2-dimensional for any $z \notin -\mathbb{N}$.

Now, consider the Fourier transform of a homogeneous distribution u. By definition if $\phi \in \mathcal{S}(\mathbb{R})$ and $a > 0$,

$$(\hat{\phi})(\frac{\xi}{a}) = \int e^{-ix\xi/a}\phi(x)dx = a\int e^{-it\xi}\phi(at)dt = a\mathcal{F}\phi(a\cdot). \tag{12}$$

Thus from the definition of the Fourier transform of distributions and the homogeneity of u,

$$\hat{u}(\phi(\frac{\cdot}{a})) = u(\mathcal{F}(\phi(\frac{\cdot}{a}))) = a^{-1}u((\hat{\phi})(a\cdot)) = a^{-z-1}u(\hat{\phi}) = a^{-z-1}\hat{u}(\phi). \tag{13}$$

That is,

$$u \text{ homogeneous of degree } z \Longleftrightarrow \hat{u} \text{ homogeneous of degree } -z-1. \tag{14}$$

The arguments above for $z \notin -\mathbb{N}$ therefore also apply to show that the space of homogeneous distributions is at least two-dimensional for all $-z-1 \notin -\mathbb{N}$, that is $z \notin \mathbb{N}_0$. So indeed the argument applies for all $z \in \mathbb{C}$.

I did not do the rest of the argument in class because we are missing a couple of the intgredients.

To complete the proof that the space of homogeneous distributions is exactly two dimensional we need to localize away from zero, Since $\phi(x/a)$ depends smoothly on a as an element of $\mathcal{S}(\mathbb{R})$ it follows that we can differentiate the homogeneity identity (11) and conclude that Euler's identity holds

$$x\frac{du}{dx} - zu = 0. \tag{15}$$

In $x >, 0$ x^z is smooth so the product $x^{-z}u$ is well-defined and satisfies

$$\frac{d}{dx}(x^{-z}u) = 0 \Longrightarrow u = cx^z \text{ in } x > 0 \tag{16}$$

for some constant c. Since $x_+^z = x^z$ in $x > 0$, a similar argument in $x < 0$ shows that provided $z \notin -\mathbb{N}$ (so that $x_\pm^z$ are well-defined) there are constant c, d such that

$$\operatorname{supp}(u - cx_+^z - dx_-^x) \subset \{0\}. \tag{17}$$

It follows that $u - cx_+^z - dx_-^x$ is a finite sum of derivatives of δ. The kth derivative has homogeneity $-k-1$, so by scaling it follows that

$$u = cxz_+ + dx_-^z,\ z \notin -\mathbb{N}. \tag{18}$$

Thus the space of homogeneous distributions is two dimensional in these cases. The Fourier transform argument above shows that this must be true in general. □

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