Read: Notes Chapter 3, Sections 4 and 5.

BEFORE LECTURE

In lecture 3 we showed that $\mathcal{S}(\mathbb{R}^n)$ (in fact $\mathcal{C}_c^\infty(\mathbb{R}^n)$) is dense in $L^2(\mathbb{R}^n)$ and using that and the identity

$$\hat{u} \hat{v} = (2\pi)^n \int u \bar{v}, \; u, \; v \in \mathcal{S}(\mathbb{R}^n)$$

we concluded that the Fourier transform extends by continuity to an (essentially isometric) isomorphism $F : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$. Make sure you understand why this is also the restriction of the map we had previously defined $F : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$.

- $(1 + |x|^2)^{s/2}$ is a multiplier on $\mathcal{S}(\mathbb{R}^n)$ and hence $\mathcal{S}'(\mathbb{R}^n)$.
- Sobolev spaces, $H^s(\mathbb{R}^n) \ni u$ iff $u \in \mathcal{S}'(\mathbb{R}^n)$ and $(1 + |\xi|^2)^{s/2} \hat{u} \in L^2(\mathbb{R}^n)$.
- Density of $\mathcal{S}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$.
- Sobolev spaces of positive integral order.
- (Didn’t do this) Sobolev spaces of negative integral order.
- Sobolev embedding $H^s(\mathbb{R}^n) \subset C_0^\infty(\mathbb{R}^n)$ if $s > n/2$.
- Sobolev spaces of fractional order.

AFTER LECTURE

The characterization of the condition $u \in H^s(\mathbb{R}^n)$ for $u \in L^2(\mathbb{R}^n)$ and $0 < s < 1$ is not in the notes (I think). Here is more-or-less what I did in class today.

We know that if $s > 0$ and $k$ is the integral part of $s$ then $u \in H^s(\mathbb{R}^n)$ is equivalent to the statement that $D^k u \in H^{s-k}(\mathbb{R}^n)$, $|\alpha| \leq k$. So we can concentrate on the case $s \in (0, 1)$.

**Proposition 1.** If $0 < s < 1$ then $u \in H^s(\mathbb{R}^n)$ if and only if $u \in L^2(\mathbb{R}^n)$ and

$$\int \int \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy < \infty. \quad (1)$$

*Proof.* If $u \in L^2(\mathbb{R}^n)$ the integrand in (1) is a non-negative measurable function so the finiteness of the integral is a well-defined condition. In fact the part of the integral away from the diagonal, $x = y$, is already finite — if $c > 0$ then

$$\int \int_{|x - y| > c} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy < \infty. \quad (2)$$
To see this use the inequality, \(|u(x) - u(y)|^2 \leq 2|u(x)|^2 + 2|u(y)|^2\) giving two integrals which are the same, so that after changing variables and using Fubini’s theorem

\[
(3) \quad \iint_{|x-y|>c} \frac{|u(x)|^2}{|x-y|^{n+2s}} \, dxdy = \int |u(x)|^2 \, dx \int_{|z|>c} |z|^{-n-2s} \, dz
\]

where both factors are finite. Thus the significance of \((1)\) is in the convergence across the diagonal.

Now, if \(u \in \mathcal{S}(\mathbb{R}^n)\) then \((1)\) does indeed hold. We have just seen the convergence when \(|x-y| > c\) and in \(|x-y| < c\) Taylor’s formula (or the mean value theorem) gives, in view of the rapid decay of the derivative

\[
(4) \quad |u(x) - u(y)| \leq C|x-y|(1+|x|)^{-n}, \quad |x-y| \leq c
\]

so this part of the integral is also finite

\[
(5) \quad \iint_{|x-y|<c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy \leq C \int (1+|x|)^{-2n} \, dx \int_{|z|<c} |z|^{-n-2s+2} \, dz
\]

and since the power of \(|z|\) is strictly larger than \(-n\) the integral converges across \(|z| = 0\).

So, now consider the integral \((1)\) when \(u \in \mathcal{S}(\mathbb{R}^n)\); we have just seen that it is a well-defined Lebesgue integral. We can change variable to give, again by Fubini

\[
\text{(which tells us that the first integral converges a.e. and the result is integrable)} \quad \int dz \, |z|^{-n-2s} \int |u(z+y) - u(y)|^2 \, dy.
\]

Then we use Plancherel’s formula on the inner integral to write it as

\[
(6) \quad \int |u(z+y) - u(y)|^2 \, dy = (2\pi)^{-n} \int |\mathcal{F}(u(z+\cdot) - u(\cdot))|^2 \, d\xi,
\]

\[
\mathcal{F}(u(z+\cdot) - u(\cdot))(\xi) = (e^{iz\xi} - 1)\hat{u}(\xi) \implies \int dz \, |z|^{-n-2s} \int |u(z+y) - u(y)|^2 \, dy = \int d\xi |\mathcal{F}(\hat{u})(\xi)|^2, \quad \mathcal{F}(\xi) = \int |e^{iz\xi} - 1|^2 \, dz.
\]

As it must by Fubini’s theorem, the integrand defining \(F(\xi)\) does indeed converge. Near infinity the integrand is bounded by \(2|z|^{-n-2s}\) which is integrable and near zero, by Taylor’s formula, it is bounded by \(C|z|^{-n-2s+2}\) which is also integrable. Furthermore it is clearly rotation-invariant. Applying an orthogonal transformation \(F(\Omega \xi) = F(\xi)\) using the change of variable to \(O^\top z\). Thus in fact \(F(\xi) = F(|\xi|)\). It is also homogeneous of degree \(2s\) as can be seen by scaling the variable. Thus in fact

\[
(7) \quad F(\xi) = c|\xi|^{2s}, \quad c > 0.
\]

So in fact for \(u \in \mathcal{S}(\mathbb{R}^n),\)

\[
(8) \quad \iint \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy = c \int |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi.
\]

Since \(1 + |\xi|^{2s}\) is bounded above and below by positive multiples of \((1 + |\xi|^2)^s\)

\[
\left(\|u\|_{L^2}^2 + \iint \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy\right)^{\frac{1}{2}}
\]

is a Hilbert norm which is equivalent to the \(H^n\) norm on \(\mathcal{S}(\mathbb{R}^n)\).
So this proves the result: the density of $S(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$ means that if $u \in H^s(\mathbb{R}^n)$ then we can find a sequence $u_n \in S(\mathbb{R}^n)$ such that $u_n \to u$ in $L^2(\mathbb{R}^n)$ and $u_n$ converges in $H^s(\mathbb{R}^n)$ (to $u$ of course). This implies the convergence of the integral $(1)$ for $u_n$ as $n \to \infty$ and hence that the integrals for $u$ over $|x-y| > \delta$ are bounded by a fixed constant independent of $\delta > 0$. This, by monotone convergence, implies that the integral for $u$ is finite and conversely, and that $(8)$ holds in the limit. ☐

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