18.155 LECTURE 4, 2015

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ABSTRACT. Notes before and after lecture – if you have questions, ask!

Read: Notes Chapter 3, Sections 4 and 5.

Before lecture

In lecture 3 we showed that $\mathcal{S}(\mathbb{R}^n)$ (in fact $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$) is dense in $L^2(\mathbb{R}^n)$ and using that and the identity

$$\int \hat{u}\overline{\hat{v}} = (2\pi)^n \int u\overline{v},\ u,\ v \in \mathcal{S}(\mathbb{R}^n)$$

we concluded that the Fourier transform extends by continuity to an (essentially isometric) isomorphism $\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$. Make sure you understand why this is also the restriction of the map we had previously defined $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$.

- $(1 + |x|^2)^{s/2}$ is a multiplier on $\mathcal{S}(\mathbb{R}^n)$ and hence $\mathcal{S}'(\mathbb{R}^n)$.
- Sobolev spaces, $H^s(\mathbb{R}^n) \ni u$ iff $u \in \mathcal{S}'(\mathbb{R}^n)$ and $(1 + |\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)$.
- Density of $\mathcal{S}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$.
- Sobolev spaces of positive integral order.
- (Didn't do this) Sobolev spaces of negative integral order.
- Sobolev embedding $H^s(\mathbb{R}^n) \subset \mathcal{C}^0_0(\mathbb{R}^n)$ if $s > n/2$.
- Sobolev spaces of fractional order.

After lecture

The characterization of the condition $u \in H^s(\mathbb{R}^n)$ for $u \in L^2(\mathbb{R}^n)$ and $0 < s < 1$ is not in the notes (I think). Here is more-or-less what I did in class today.

We know that if $s > 0$ and k is the integral part of s then $u \in H^s(\mathbb{R}^n)$ is equivalent to the statement that $D^\alpha u \in H^{s-k}(\mathbb{R}^n)$, $|\alpha| \le k$. So we can concentrate on the case $s \in (0, 1)$.

Proposition 1. *If $0 < s < 1$ then $u \in H^s(\mathbb{R}^n)$ if and only if $u \in L^2(\mathbb{R}^n)$ and*

$$\iint \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy < \infty. \tag{1}$$

Proof. If $u \in L^2(\mathbb{R}^n)$ the integrand in (1) is a non-negative measurable function so the finiteness of the integral is a well-defined condition. In fact the part of the integral away from the diagonal, $x = y$, is already finite – if $c > 0$ then

$$\iint_{|x-y|>c} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy < \infty. \tag{2}$$

To see this use the inequality, $|u(x)-u(y)|^2 \le 2|u(x)|^2+2|u(y|^2$ giving two integrals which are the same, so that after changing variables and using Fubini's theorem

$$\iint_{|x-y|>c} \frac{|u(x)|^2}{|x-y|^{n+2s}} dxdy = \int |u(x)|^2 dx \int_{|z|>c} |z|^{-n-2s} dz \tag{3}$$

where both factors are finite. Thus the significance of (1) is in the convergence across the diagonal.

Now, if $u \in \mathcal{S}(\mathbb{R}^n)$ then (1) does indeed hold. We have just seen the convergence when $|x-y|>c$ and in $|x-y|<c$ Taylor's formula (or the mean value theorem) gives, in view of the rapid decay of the derivative

$$|u(x)-u(y)| \le C|x-y|(1+|x|)^{-n},\ |x-y| \le c \tag{4}$$

so this part of the integral is also finite

$$\iint_{|x-y|<c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \le C\int (1+|x|)^{-2n} dx \int_{|z|<c} |z|^{-n-2s+2} dz \tag{5}$$

and since the power of $|z|$ is strictly larger than $-n$ the integral converges across $|z|=0$.

So, now consider the integral (1) when $u \in \mathcal{S}(\mathbb{R}^n)$; we have just seen that it is a well-defined Lebesgue integral. We can change variable to give, again by Fubini (which tells us that the first integral converges a.e. and the result is integrable)

$$\int dz|z|^{-n-2s} \int |u(z+y)-u(y)|^2 dy.$$

Then we use Plancherels' formula on the inner integral to write it as

(6)

$$\int |u(z+y)-u(y)|^2 dy = (2\pi)^{-n} \int |\mathcal{F}(u(z+\cdot)-u(\cdot))|^2 d\xi,$$
$$\mathcal{F}(u(z+\cdot)-u(\cdot))(\xi) = (e^{z\cdot\xi}-1)\hat{u}(\xi) \Longrightarrow$$
$$\int dz|z|^{-n-2s} \int |u(z+y)-u(y)|^2 dy = \int d\xi F(\xi)|\hat{u}(\xi)|^2,\ F(\xi) = \int \frac{|e^{iz\cdot\xi}-1|^2}{|z|^{n+2s}} dz.$$

As it must by Fubini's theorem, the integrand defining $F(\xi)$ does indeed converge. Near infinity the integrand is bounded by $2|z|^{-n-2s}$ which is integrable and near zero, by Taylor's formula, it is bounded by $C|z|^{-n-2s+2}$ which is also integrable. Furthermore it is clearly rotation-invariant. Applying an orthogonal transformation $F(O\xi) = F(\xi)$ using the change of variable to $O^t z$. Thus in fact $F(\xi) = F(|\xi|)$. It is also homogeneous of degree $2s$ as can be seen by scaling the variable. Thus in fact

$$F(\xi) = c|\xi|^{2s},\ c>0. \tag{7}$$

So in fact for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\iint \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy = c\int |\xi|^{2s}|\hat{u}(\xi)|^2 d\xi. \tag{8}$$

Since $1+|\xi|^{2s}$ is bounded above and below by positive multiples of $(1+|\xi|^2)^s$

$$\left(\|u\|_{L^2}^2 + \iint \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy\right)^{\frac{1}{2}}$$

is a Hilbert norm which is equivalent to the H^s norm on $\mathcal{S}(\mathbb{R}^n)$.

So this proves the result; the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$ means that if $u \in H^s(\mathbb{R}^n)$ then we can find a sequence $u_n \in \mathcal{S}(\mathbb{R}^n)$ such that $u_n \to u$ in $L^2(\mathbb{R}^n)$ and u_n converges in $H^s(\mathbb{R}^n)$ (to u of course). This implies the convergence of the integral (1) for u_n as $n \to \infty$ and hence that the integrals for u over $|x-y| > \delta$ are bounded by a fixed constant independent of $\delta > 0$. This, by monotone convergence, implies that the integral for u is finite and conversely, and that (8) holds in the limit. □

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