# 18.155 LECTURE 3, 15 SEPTEMBER 2016

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ABSTRACT. Notes before and then after lecture.

Read: Notes Chapter 3, Section 2 and first part of Section 3.

Central result for today is that $\sqrt{2\pi^n}\mathcal{F}$ extends by continuity and density from $\mathcal{S}$ to a unitary operator on $L^2(\mathbb{R}^n)$ and that this allows us to define the Sobolev spaces $H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ for each $s \in \mathbb{R}$ as the inverse Fourier transforms of the weighted $L^2$ spaces $(1+|\xi|^2)^{-s/2}L^2(\mathbb{R}^n)$.

## BEFORE LECTURE

- Tempered distributions and operations.
  Differentiation, multiplication by functions of slow growth, tranlation, Fourier tranform, relations.
- Compactly supported smooth functions – $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ consisting of the functions which vanish outside some compact set.
  Bump functions.
  Topology – think about it, don't worry!
- Density of $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$.
- Convolution
  $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) * L^1_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$.
- Density of test functions in square-integrable functions
- Fourier transform of square-integrable functions
- Sobolev spaces
  I got to about here.
- Sobolev spaces of integral order
- Density of $\mathcal{S}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$.
- Sobolev embedding (probably will not get this far).

## AFTER LECTURE

- I talked in a slightly abstract way (which is not in the notes I think) about the transfer of operations on $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, the latter being the dual of the former. We embed $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ using the (complex linear, not sesquilinear) integral pairing

$$\langle \phi, \psi \rangle = \int \phi\psi,\ \phi,\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{1}$$

  On the other hand we have the 'duality pairing'

$$\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (u, \phi) \longmapsto u(\phi) \in \mathbb{C} \tag{2}$$

  and we are using (1) to define

$$U_\phi(\psi) = \langle \phi, \psi \rangle. \tag{3}$$

When we show that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$

[You might like to work out a proof for yourself along the following lines. By density we mean 'in the weak topology', so all we need to show is that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then there exists a sequence $u_n \in \mathcal{S}(\mathbb{R}^n)$ such that $\langle u_n, \psi\rangle \longrightarrow u(\psi)$ for each $\psi \in \mathcal{S}(\mathbb{R}^n)$ – no uniformity. To do this you can use our sequence of cutoffs (from later in this lecture ....) $\mu_k(x) = \mu(x/k)$ where $\mu \in \mathcal{C}^\infty_{\text{c}}(\mathbb{R}^n)$ has $\mu(x) = 1$ in $|x| < \frac{1}{2}$. If you work at it a bit you can see that $u_k = \mathcal{G}(\mu_k(\xi)\mathcal{F}(\mu_k(x)u)$ is a sequence in $\mathcal{C}^\infty_{\text{c}}(\mathbb{R}^n)$ with the desired properties, we will do this later.]

Anyway, the operations we consider $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ are continuous linear maps which have transposes $A^t : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ also continuous linear maps, such that

$$\langle A\phi, \psi\rangle = \langle \phi, A^t\psi\rangle,\ \forall\ \phi,\ \psi \in \mathcal{S}(\mathbb{R}^n) \tag{4}$$

then we just *define*

$$A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ by } Au(\psi) = u(A^t\psi). \tag{5}$$

As the composite $u \circ A^t\ Au$ is then continuous, so defines a map as indicated.

We have done this for $A = \partial_j$, $A^t = -A$, $A = \times f$ where $f$ is of slow growth (e.g. a polynomial) and then $A^t = A$. Also $A = T_y$, translation by $y \in \mathbb{R}^n$ and then $A^t = T_{-y}$ (the inverse of $T_y$) and most importantly $A = \mathcal{F} = A^t$. Convolution with an element of $\mathcal{S}(\mathbb{R}^n)$ will give another example.

I said in lecture that there are continuous linear maps $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ which do not have 'formal' transposes acting on $\mathcal{S}(\mathbb{R}^n$.; One such example is a finite rank operator

$$A\phi = \sum_j v_j(\phi)\psi_j,\ v_j \in \mathcal{S}'(\mathbb{R}^n),\ \psi_j \in \mathcal{S}(\mathbb{R}^n). \tag{6}$$

The formal transpose of this (assuming $v_j \notin \mathcal{S}(\mathbb{R}^n)$ maps $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, as it must, but does not map $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

- To prove the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ I observed from standard properties of the Lebesgue integral that $\mathcal{C}^0_{\text{c}}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is dense, so it sufficies to prove that any continuous function of compact support can be approximated uniformly by a sequence in

$$\mathcal{C}^\infty_{\text{c}}(\mathbb{R}^n) = \{\phi \in \mathcal{S}(\mathbb{R}^n); \exists\ R,\ \phi(x) = 0 \text{ in } |x| > R\}. \tag{7}$$

[Really we wan the sequuence to vanish outside a fixed ball but this could be arranged afterwards anyway].

- I did this using the convolution integral. If $u,\ v \in \mathcal{C}^0_{\text{c}}(\mathbb{R}^n)$ both vanish in $|x| > R$ then

$$u * v(x) = \int u(x-y)v(y)du = \int u(y)v(x-y)dy = v * u(x) \in \mathcal{C}^0_{\text{c}}(\mathbb{R}^n) \tag{8}$$

is a well-defined Riemann integral and vanishes when $|x| > R$.

The continuity in $x$ follows by direct estimation

$$(9)\quad |u*v(x+z)-u*v(x)| = |\int (u(x+z-y)-u(x-y)\,v(y)| \\ \leq \sup|v| \sup_{|y|\leq R} |v(x+z-y)-v(x-y)|$$

and the *uniform* continuity of $v$ shows the right side tends to zero as $|z|\to 0$.

- If $v\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $u\in\mathcal{C}^0_{\mathrm{c}}(\mathbb{R}^n)$ a similar argument shows that $U*v\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Namely to prove on derivative exists and is continuous compute the difference quotient using the second version of the convolution

$$(10)\quad \frac{u*v(x+te_j)-u*v(x)}{t} = \int u(y)\frac{u(x+te_j-y)-u(x-y))}{t}$$

and using the FTC or intermediate value theorem to show that the integrand on the right converges uniformly to the derivative. So in fact

$$(11)\quad \partial_j u*v = u*(\partial_j v)$$

so we can immediately argue by induction that $U*v\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ if $v\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

- I also proved the density directly by a standard argument using an 'approximate identity' (for convolution). Namely if $\phi\geq 0$, $\phi(x)=0$ in $|x|>1$, $\phi\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $\int\phi=1$ (which we can easily arrange with a bump function) then $\phi_k(x)=k^{-n}\phi(kx)\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and

$$(12)\quad u*\phi_k\to u \text{ uniformly if } u\in\mathcal{C}^0_{\mathrm{c}}(\mathbb{R}^n).$$

The proof is to write the difference 'cleverly' using the fact that the normalization is chosen so $\int\phi_k=1$ for all $k$ :

$$(13)\quad u*\phi_k(x)-u(x) = \int (u(x-y)-u(x))\,\phi_k(y)dy.$$

Then the difference can be estimated using the positivity of $\phi$ and the fact that $\phi_k(y)$ vanishes in $|y|>1/k$

$$(14)\quad |u*\phi_k(x)-u(x)| \leq \int |u(x-y)-u(x)|\phi_k(y)dy \leq \sup_{|y|\leq 1/k}|u(x-y)-u(x)|$$

and the uniform continuity of $u$.

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