18.155 LECTURE 21
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Abstract. Notes before and then after lecture.

I decided I would spend the four remaining lectures discussing some aspects of analysis on manifolds – in particular the Laplace-Beltrami operator and Hodge Laplacian. Originally I had planned to talk about scattering theory but Semyon Dyatlov will be devote much of 18.156 to this next semester.

I assume you know about manifolds but let me start from the beginning *in principle* so that we agree on notation.

Before lecture

- A diffeomorphisms of open sets in Euclidean space; $F : \Omega \longrightarrow \Omega'$, $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^{n'}$ open is a smooth map which is a bijection with a smooth inverse, necessarily $n = n'$ if the sets are non-empty. Equivalently $F^*u = u \circ F$ induces a bijection

$$F^* : \mathcal{C}^\infty(\Omega') \longrightarrow \mathcal{C}^\infty(\Omega) \text{ or} \\ F^* : \mathcal{C}^\infty_{\mathrm{c}}(\Omega') \longrightarrow \mathcal{C}^\infty_{\mathrm{c}}(\Omega). \tag{L22.1}$$

- We need to understand the behaviour of various functionals and spaces under diffeomorphisms. The most basic of these is the integral for which the transformation property is well-known.

Proposition 1. *If $F : \Omega \longrightarrow \Omega'$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ and $u \in \mathcal{C}^\infty_c(\Omega')$ then*

$$\int_\Omega F^*u(x)|J(x)|dx = \int_{\Omega'} u(y)dy,\ J(x) = \det \frac{\partial F_i(x)}{\partial x_j}. \tag{L22.2}$$

The presence of this 'Jacobian factor' in the integral is the reason that the integral of a compactly supported smooth function on a manifold is *not* invariantly defined.

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Definition 1. A smooth manifold is a metrizable topological space M (connected unless stated explicitly otherwise) with a given space $\mathcal{F}(M) \subset \mathcal{C}^0(M)$ of 'smooth functions' where

(1) M has a covering by open sets U_α for each of which there are elements $x_j \in \mathcal{F}(M)$, $j = 1, \dots, n$ such that $F : U_\alpha \ni p \longmapsto (x_1(p), \dots, x_n(p)) \in \mathbb{R}^n$ is a homeomorphism to an open set $U'_\alpha \subset \mathbb{R}^n$ and

$F^* : \mathcal{C}^\infty_{\mathrm{c}}(U'_\alpha) \longrightarrow \mathcal{C}^0(M)$ has range precisely

$$\{u \in \mathcal{F}(M); u = 0 \text{ on } M \setminus K,\ K \Subset U_\alpha\}.$$

(2) $\mathcal{F}(M)$ has the sheaf property that for any open covering U_α of M if $u \in \mathcal{C}^0(M)$ and for each α there exists $v_\alpha \in \mathcal{F}(M)$ such that $u = v_\alpha$ on U_α then $u \in \mathcal{F}(M)$.

We then write $\mathcal{C}^\infty(M) = \mathcal{F}(M)$; if the second condition fails simply define $\mathcal{C}^\infty(M)$ by this condition for an open cover by coordinate patches as in the first condition and check that the definition then holds. Thus the second part is a 'maximality' condition on $\mathcal{C}^\infty(M)$.

- The standard definition.
- Examples include (connected) open subsets of $\mathbb{R}^n$, spheres and other embedded submanifolds of $\mathbb{R}^N$ and quotients such as the torus.
- Today I want to get as far as defining the analogues of spaces we have talked about

$$\begin{tikzcd}
\mathcal{C}^\infty_{\mathrm{c}}(M) \arrow[r] \arrow[d] & \mathcal{C}^\infty(M) \arrow[d] & s \ge s' & \mathcal{C}^\infty(M) \arrow[d] \\
H^s_{\mathrm{c}}(M) \arrow[r] \arrow[d] & H^s_{\mathrm{loc}}(M) \arrow[d] & & H^s(M) \arrow[d] \\
H^{s'}_{\mathrm{c}}(M) \arrow[r] \arrow[d] & H^{s'}_{\mathrm{loc}}(M) \arrow[d] & & H^{s'}(M) \arrow[d] \\
\mathcal{C}^{-\infty}_{\mathrm{c}}(M) \arrow[r] & \mathcal{C}^{-\infty}(M) & & \mathcal{C}^{-\infty}(M)
\end{tikzcd} \tag{L22.3}$$

where the second column is for M compact and all arrows are dense injections.

- In principle this is easy – we just identify the spaces locally. For $s \ge 0$ the resulting objects are functions by for $s < 0$ may not be so. We could still define abstract sheaves but we really want the duality idea that we started with and that leads us to define (and explain)

$$\mathcal{C}^{-\infty}(M; V) = \mathcal{C}^\infty_{\mathrm{c}}(M; V' \otimes \Omega)',\ \mathcal{C}^{-\infty}_{\mathrm{c}}(M; V) = \mathcal{C}^\infty(M; V' \otimes \Omega)' \tag{L22.4}$$

for any vector bundle V over M.

- The core point here is the existence of an invariantly-defined integral; this is what is behind (L22.4):

$$\int_M : \mathcal{C}^\infty_{\mathrm{c}}(M; \Omega) \longrightarrow \mathbb{C}. \tag{L22.5}$$

- There are many manifolds which are 'functorially associated' to a given manifold M. The primary ones are the tangent and cotangent bundles. As sets these are unions over M of vector spaces

(L22.6)
$$TM = \bigcap_{p \in M} T_pM,\ T_pM = \{v : \mathcal{C}^\infty(M; \mathbb{R}) \longrightarrow \mathbb{R}; v(fg) = f(p)v(g) + g(p)v(f)\}$$
$$T^*M = \bigcap_{p \in M} T^*_pM,\ T^*_pM = \mathcal{I}_p/\mathcal{I}_p^2,\ \mathcal{I}_p = \{v \in \mathcal{C}^\infty(M; \mathbb{R}); v(p) = 0\},\ \mathcal{I}_p^2 = \mathrm{sp}\{fg; f,\ g \in \mathcal{I}_p\}.$$

The tangent space T_pM is the space of derivations on $\mathcal{C}^\infty(M)$ at p. Observe that there is a pairing

$$T_pM \times T_p^*M \ni (v,[f]) \longmapsto vf \in \mathbb{R}. \tag{L22.7}$$

This is a 'perfect pairing' identifying each as the dual of the other.

- Vector bundles, densities and distributions.
- Operators

After lecture

(1) Here is a proof that either of the conditions in (L22.1) is equivalent to F being a diffeomorphism.

That F being a diffeomorphism implies (L22.1) is straightforward – the pull-back under a smooth map $F : \Omega \longrightarrow \Omega'$ always defines a linear map

$$F^* : \mathcal{C}^\infty(\Omega') \longrightarrow \mathcal{C}^\infty(\Omega), \tag{L22.8}$$

by the chain rule. Then the inverse $G = F^{-1}$ defines a linear map $G^* : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega')$ which is a 2-sided inverse to F^*. Since F is a homeomorphism $F^{-1}(K)$ is compact if $K \Subset \Omega'$ (since it is $G(K)$) and so the second part of (L22.1) follows from the first part when F is a diffeomorphism.

Conversely, if $F : \Omega \longrightarrow \Omega'$ is a map such that F^* defines a bijection as in the first part of (L22.1) then, since the components of $F = (F_1, \dots, F_n)$ are the pull-backs of the coordinate functions y_i on Ω' it follows that F is smooth. Similarly the restrictions to Ω of the coordinate functions x_j on $\mathbb{R}^n$ are elements of $\mathcal{C}^\infty(\Omega)$ and so of the form $F^*g_j = g_j \circ F$ for some elements $g_j \in \mathcal{C}^\infty(\Omega')$. Thus $G(y) = (g_1(y), \dots, g_n(y))$ defines a smooth map $G : \Omega' \longrightarrow \mathbb{R}^n$ such that $G \circ F(x) = ((F^*g_1)(x), \dots, (F^*g_n)(x)) = (x_1, \dots, x_n)\big|_\Omega$, i.e. $G \circ F = \mathrm{Id}_\Omega$ so $G : \Omega' \longrightarrow \Omega$ is a left inverse of F. It follows that $F^* \circ G^* = \mathrm{Id}$ so G^* is a right inverse of F^* as a linear map and since F^* is a bijection G^* is the two-sided inverse. The same argument with variables reversed shows that G is a two-sided inverse of F which is therefore a diffeomorphism.

If the second version of (L22.1) is assumed instead of the first it follows directly that $F : \Omega \longrightarrow \Omega'$ is proper. More precisely, if K_j is an exhaustion by compact sets of Ω', with $K_{j+1} \subset \operatorname{int} K_j$ then there is a corresponding sequence $\chi_j \in \mathcal{C}^\infty_{\mathrm{c}}(\operatorname{int} K_j)$ with $\chi_j = 1$ on K_{j-1}. Since $F^*\chi_j \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ it follows that $f^{-1}(K_j)$ is a compact exhaustion of Ω. Now a function u on Ω' is in $\mathcal{C}^\infty(\Omega')$ if and only if it is equal to some element $v_j \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega')$ on each K_j. It follows that $F^*u \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ and conversely. Thus the second version of (L22.1) implies the first.

(2) Now, consider the identity (L22.2) which is of course a well-known result from measure theory (and the regularity hypotheses on F can be weakened). However, just because we can, let's use distribution theory to prove it.

Since we know that if F is a diffeomorphism then F^* is a *continuous* bijection as in (L22.1) we can consider the integral over Ω' as a functional

$$I : \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \ni v \longrightarrow \int_{\Omega'} G^* v dy \in \mathbb{C} \Longrightarrow \\ \exists\ I \in \mathcal{C}^{-\infty}(\Omega) \text{ s.t. } \int_{\Omega'} G^* v = I(v) = \text{`} \int I(x)v(x)dx'. \tag{L22.9}$$

So to prove (L22.2) we only need to show that I actually is the smooth function $|J(x)|$.

We can see directly that the functional U is bounded by the supremum norm of v for $\mathrm{supp}(v) \subset K \Subset \Omega$ fixed, since

$$\sup |G^* v| = \sup |v|.$$

Since this is a stronger norm than L^2 this implies that $I \in L^2_{\mathrm{loc}}(\Omega)$ by Riesz' Representation Theorem. Mover, we can 'compute' the derivatives of U since

$$\partial_j I(v) = -I(-\partial_j v) = -\int_{\Omega'} G^*(\partial_j v), \\ G^*(\partial_j v) = (\partial_j v)(g(y)) = \sum_{i=1}^{j} w_{ji}(y)\partial_{y_j}(G^* v(y)),\ w_{ji}(x) = (\partial_i g_j(y))^{-1} \tag{L22.10}$$

by the chain rule where we use the invertibility of the Jacobian matrix. Integrating by parts in the integral it follows that $\partial_j I \in L^2_{\mathrm{loc}}(\Omega)$ as well. Iterating the argument for higher derivatives shows that indeed $I \in \mathcal{C}^\infty(\Omega)$.

So now it follows that the distribution I extends by continuity to all $v \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega)$. Consider what happens then for $v = \delta_{\bar{x}}$, the Dirac delta at some point $\bar{x} \in \Omega$. We can use the limit $\delta_{\bar{x}} = \lim_{\epsilon \downarrow 0} \epsilon^{-n}\chi((x - \bar{x})/\epsilon)$ for some bump function of integral one. Then

$$G^* \left(\epsilon^{-n}\chi((\cdot - \bar{x})/\epsilon)\right)(y) = \epsilon^{-n}\chi((g(y) - g(\bar{y}))/\epsilon),\ f(\bar{x}) = \bar{y}. \tag{L22.11}$$

Exercise 1. Show (just using the behaviour of the Riemann integral under linear transformations) that as a sequence of compactly supported distribution on Ω',

$$\int \epsilon^{-n}\chi((g(y) - g(\bar{y}))/\epsilon) \to |\det \frac{\partial g_j}{\partial y_i}|^{-1}(\bar{y}). \tag{L22.12}$$

From this it follows that $I(\bar{x}) = |\det \frac{\partial g_j}{\partial y_i}|^{-1}(\bar{y}) = |J(\bar{x})|$ and this in turn proves (L22.2) – which you knew anyway.

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