18.155 LECTURE 20
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Abstract. Notes before and then after lecture.

Read: See Hörmander's treatise, Chapter 12 (Vol 2) if you want more detail.

Before lecture

- Paley-Wiener again
- Hyperbolic polynomials
- The forward fundamental solution – outline
- Uniqueness for the (tempered) Cauchy problem
- Forward fundamental solution for the wave operator

Additonal notes before lecture

- Paley-Wiener again:

$$\{u \in L^2(\mathbb{R}); u = 0 \text{ in } x < 0\} \simeq \{\hat{u} : \{(\xi + i\eta) \in \mathbb{C}; \eta < 0\} \longrightarrow \mathbb{C}; \\ \text{holomorphic and } \sup_{0>\eta>-\infty} \int |\hat{u}(\xi + i\eta)|^2 d\xi < \infty\}. \tag{1}$$

also

$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}) \simeq \{\hat{u} : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire and for some } A, \\ \sup(1 + |\xi|^2)^N \exp(-A|\eta|)|\hat{u}(\xi + i\eta)| < \infty \ \forall\ N. \tag{2}$$

$$\{u \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}); \operatorname{supp}(u) \subset (-\infty, 0] \simeq \{\hat{u} : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire and for some } A > 0, \\ \sup(1 + |\xi|^2)^N \exp(-A(\eta)_+)|\hat{u}(\xi + i\eta)| < \infty\}. \tag{3}$$

The proofs of the forward versions of these results are pretty much the same. For instance if $u \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ then the Fourier-Laplace transform

$$\hat{u}(\xi + i\eta) = \int e^{-ix(\xi+i\eta)} u(x)dx = \mathcal{F}(e^{x\eta}u) \tag{4}$$

is defined for all $\xi + i\eta \in \mathbb{C}$. The norm of $e^{x\eta}u$ in H^N is bounded by $C_N \exp(A|\eta))$ provided the support of u is contained in a ball of radius smaller than A and by differentiation under the integral sign is holomorphic, i.e. entire. This gives the map in (2). The converse follows by first observing that the inverse Fourier transform of $\hat{u}$ restricted to the real space defines an element of $H^\infty(\mathbb{R})$.

Hyperbolicity

- Hyperbolic polynomials. The model for these corresponds to the wave operator
$$D_t^2 - \Delta_x = P(D_t, D_x),\ \ P(\tau, \xi) = \tau^2 - |\xi|^2 \text{ on } \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}.$$
- We say that a polynomial of degree m in n variables is hyperbolic with respect to the (co)-direction $N \in \mathbb{R}^n$ if
  (1) The principal part satisfies $P_m(N) \neq 0$ ($P(D)$ is then said to be non-characteristic for the hypersurface $x \cdot N = 0$ or vice versa).
  (2) The roots s of $P(\xi + sN) = 0$ have imaginary part bounded independent of $\xi \in \mathbb{R}^n$, $|\operatorname{Im} s| < s_0$.

  If course we can replace ξ by $\xi + (\operatorname{Re} s)N$ and so think of this last condition as saying that $P(\xi + i\tau N) \neq 0$ for $|\tau| \geq s_0$.
- The first condition means that $P(\xi + sN)$ is a polynomial of degree exactly m in s for each ξ since the coefficient of s^m is $P_m(N)$.
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Lemma 1. *If P is hyperbolic with respect to N then so is P_m.*

Proof. Add another scaling variable $u > 0$ and look at $P(u^{-1}\xi + su^{-1}N) = 0$. The roots of this as a polynomial must have $|\operatorname{Im} s| < s_0 u$. The roots of this polynomial are the same as those of

$$P(s, \xi, u) = u^m P(u^{-1}\xi + su^{-1}N) = P_m(N)s^m + \sum_{j<m} q_j(u, \xi)s^j \tag{5}$$

where the coefficients are polynomials in u and ξ so in particular are continuous. Since the leading term is constant the zeros are (collectively) continuous and converge as $u \downarrow 0$ to those of

$$P(s, xi, 0) = P_m(\xi + sN) \tag{6}$$

which must therefore have only real zeros s. □

It is not the case that P_m hyperbolic implies P is hyperbolic − this is discussed in detail in Chapter 12 which is in volume two of Hörmander's treatise.

- If $\xi = N$ in (6) then the roots, s, are all equal to -1 so it follows that the roots are negative for $\xi = N'$ near N. In fact the set of $\xi = N' \in \mathbb{R}^n$ for which the zeros s of $P_m(N' + sN) = 0$ are negative is an open cone which is also convex, denote it by $\Gamma(P, N) \subset \mathbb{R}^n \setminus \{0\}$.
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Lemma 2. *If P is hyperbolic with respect to N then it is hyperbolic with respect to each $N' \in \Gamma(P, N)$.*

Proof. Since the roots of $P_m(N' + sN)$ are negative, $P_m(N') \neq 0$ and P is non-characteristic with respect to N'. We claim that roots σ of

$$P(\xi + sN + \sigma N') = 0,\ \operatorname{Im} s < s_0,\ \xi \in \mathbb{R}^n \Longrightarrow \operatorname{Im} \sigma > 0. \tag{7}$$

Hyperbolicity implies that there are no roots with σ real and as a polynomial in s this has constant leading coefficient $P_m(N') \neq 0$ so the number of roots in $\operatorname{Im} \sigma > 0$ is constant, independent of ξ and s with $\operatorname{Im} s < s_0$.

We can again scale by $u > 0$ and conclude that the number of roots with $\operatorname{Im}\sigma > 0$ of

(8) $$Q(\xi, s, u, \sigma) = u^m P(\xi + su^{-1}N + \sigma u^{-1}N') = 0,\ \xi \in \mathbb{R}^n,\ u > 0,\ \operatorname{Im} s < s_0 u$$

is constant. Again the leading term in σ is constant and the polynomial converges to $P_m(sN + \sigma N')$. The roots of this have $s/\sigma < 0$ since $N' \in \Gamma(P, N)$ and so they all have $\operatorname{Im}\sigma > 0$ which proves (7).

So now we know that $\operatorname{Im} s < s_0$ and $\operatorname{Im}\sigma < 0$ implies that $P(\xi + sN + \sigma N') \neq 0$. Thus $P(\xi + s(N + tN')) \neq 0$ if $\operatorname{Im} s < s_0$ and $t > 0$. The same argument applies with reversal of signs of imaginary parts throughout, and since $N + tN' \in \Gamma(P, N)$ it follows that P is hyperbolic with respect to $N + tN'$, $N' \in \Gamma(P, N)$. Using convexity the result follows. □

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Definition 1. A homogeneous polynomial of degree m, P_m, is said to be strictly hyperbolic with respect to N if it is non-characteristic and the roots of $P_m(\xi + sN) = 0$ are real and distinct if ξ is not a multiple of N.

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Lemma 3. *If P is strictly hyperbolic in the sense that P_m is stricly hyperbolic then P is hyperbolic.*

Proof. Certainly P_m is hyperbolic since the roots of $P_m(\xi+sN)$ must always be real, being all $-tN$ if $\xi = tN$. Suppose $\xi \perp N$ and consider the roots of

(9) $$u^m P(u^{-1}\xi + su^{-1}N) = P(\xi + sN) + u \sum_{j<m} q_j(u, \xi)s^j$$

where the coefficients are smooth in u, ξ. Since the roots at $u = 0$ are distinct they are smooth in $|u| < \epsilon$ for $|\xi| = 1$. It follows that their imaginary parts are bounded, $|\operatorname{Im} s| < Cu$. Thus for $\eta \in \mathbb{R}^n$, $\eta \perp N$ and $|\eta| > 1/\epsilon$ the imaginary parts of the roots of $P(\eta + sN)$ are uniformly bounded. In the compact region $|\eta| \leq 1/\epsilon$ they are certainly bounded so P is indeed hyperbolic. □

Maybe for Thursday, 17 November

- The most important example of a constant coefficient hyperbolic operator is the wave operator on $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_x$,

(10) $$\Box = D_t^2 - \Delta = -\partial_t^2 + \partial_{x_1}^2 + \cdots + \partial_{x_n}^2.$$

For this operator we can find a rather explicit formula for the forward fundamental solution.

- We start by looking at a holomorphic family of tempered distributions given by locally integrable functions

(11) $$G(z) = \begin{cases} (t^2 - |x|^2)^{z/2}, & t > |x| \\ 0 & \text{otherwise} \end{cases},\ \operatorname{Re}(z) > 0$$

with support in the proper cone $|t| \geq |x|$. Clearly $G(z)$ is continuous and polynomially bounded for $\operatorname{Re} z > 0$. For $\operatorname{Re}(z) > 4$ (actually 2) it is twice continuously differentiable as we can see by computing the first derivatives

(12) $$\partial_t G(z) = tG(z-2),\ \partial_{x_i} G(z) = -x_i G(z-2),$$

where these equalities hold in $t > |x|$, and observing that the rights sides are then $\mathcal{C}^1$.

- Differentiating again and collecting terms it follows that for $\operatorname{Re} z > 4$,

$$\Box G(z) = -\partial_t(tG(z-2)) - \sum_{i\geq 1} \partial_{x_i}(x_i G(z-2)) = (-z(z+n-1))G(z-2). \tag{13}$$

Shifting the variable this can be written

$$G(z) = -\frac{1}{(z+2)(z+n+1)} \Box G(z+2),\ \operatorname{Re} z > 2. \tag{14}$$

We can iterate this functional equation so that for any $k \in \mathbb{N}$

$$G(z) = \frac{(-1)^k}{(z+2)\dots(z+2k)(z+n+1)\dots(z+n+2k-1)} \Box^k G(z+2k),\ \operatorname{Re} z > 2. \tag{15}$$

- **Lemma 4.** *As a tempered distribution, $G(z)$ extends to be meromorphic in the complex plane. If $n > 1$ is odd there are simple poles at the even integers $-2, \dots, -n+1$ and double poles at the even integers $-n-1, -n-3, \dots$. If $n > 1$ is even then there are simple poles at the integers $-2p$, $p \in \mathbb{N}_0$ and $-n-1-2l$, $l \in \mathbb{N}_0$.*

  Proof. Since we know that $G(z+2k)$ is holomorphic in $\operatorname{Re} z > -2k$ the left side is meromorphic in the same region and by the uniqueness of analytic continuation the result is independent of k. The two sets of poles overlap if n is odd but not if n is even.

  So it only remains to see that there actually are poles of the claimed orders at the various points, although we are only interested in one or two of them. Look at a particular putative pole, or double pole, using the formula (15) for k very large so that the formula holds near the point in question. It follows that the residue, or double residue, is given by $\Box^p G(l)$ for some integers p and $l > 0$. Since $G(l)$ has support in $t \geq |x|$ it follows from the uniqueness theorem above, applied repeatedly – first to $\Box(\Box^{p-1}G(l))$, then to $\Box(\Box^{p-2}G(l))$ and so on – that $G(l) = 0$ as a distribution. However, by inspection this is never the case. □

- **Proposition 1.** *For n odd a multiple of the residue of $G(z)$ at $z = -n+1$ and for n even the regularized value of $G(z)$ and $z = -n+1$ is the unique forward fundamental solution of the wave operator.*

  Proof. For $l \in \mathbb{N}$ it follows from the definition that $G(l)$ is homogeneous of degree l. Using the formula (15) again, it follows that the residue at $-n-1$ in case n is even, or the double residue at $-n-1$ in case n is odd, is homogeneous of degree $-n-1$. □

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