18.155 LECTURE 16
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Abstract. Notes before and after lecture.

Read: There is something in the notes but I will approach elliptic regularity a little differently using an idea of Gårding. More details will follow.

Before lecture

- Elliptic regularity for variable coefficient operators stated.
- Symbols spaces and Fourier transform
- Commutator estimate

$$
\begin{gathered}
[b*, \times\phi] : H^s(\mathbb{R}^n) \longrightarrow H^{s+M-1}(\mathbb{R}^n),\ b = \mathcal{G}a,\ a \in S^M(\mathbb{R}^n),\ \phi \in \mathcal{S}(\mathbb{R}^n) \\
\|[b*, \times\phi]\| \le C\|a\|^{S^M}_{(k)} \|\phi\|^{\mathcal{S}}_{(k)}.
\end{gathered} \tag{1}
$$

- $\|\phi u\|_{H^s} \le C \sup|\phi| \|u\|_{H^s} + C\|\phi\|^{\mathcal{S}}_{(k)}\|u\|_{H^{s-1}}$.
- Strategies for elliptic regularity
- Regularization.

1. Several days after lecture!

Here is the main commutator result.

Proposition 1. *If $\hat{b} \in S^m(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ there exists multi-sequences sequences $\hat{b}_\alpha \in S^{m-|\alpha|}(\mathbb{R}^n)$ and $\phi_\alpha \in \mathcal{S}(\mathbb{R}^n)$, for $\alpha \in \mathbb{N}_0^n$, where $b_0 = b$, $\phi_0 = \phi$ and for any $N \in \mathbb{N}$, $k \in \mathbb{R}$ and $M \in \mathbb{R}$*

$$
\begin{gathered}
\mathcal{S}(\mathbb{R}^n) \ni u \longmapsto b * (\phi u) - \sum_{0 \le |\alpha| \le N}^{N} \phi_\alpha (b_\alpha * u) \in \mathcal{S}(\mathbb{R}^n) \\
\textit{extends by continuity to } \langle x\rangle^k H^M(\mathbb{R}^n) \longrightarrow \langle x\rangle^{-k} H^{M+m-N-1}(\mathbb{R}^n).
\end{gathered} \tag{1}
$$

The maps $S^m(\mathbb{R}^n) \ni \hat{b} \longrightarrow \widehat{b_\alpha} \in S^{m-|\alpha|}$ can be chosen to be continuous and the norm on the error term is bounded by the product of a seminorm on ϕ and a seminorm on $\hat{b}$.

Proof. We know that $b \in \mathcal{S}'(\mathbb{R}^n)$ has singular support contained in the origin and if $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ has support in $|x| < 1$ and is identically equal to one near 0 then $(1-\chi)b \in \mathcal{S}(\mathbb{R}^n)$. Moreover, if $\alpha \in \mathbb{N}_0^n$ then $\widehat{x^\alpha b} \in S^{m-|\alpha|}(\mathbb{R}^n)$.

Now we know that the Schwartz kernel of $b * \phi$ is just

$$b(x-y)\phi(y).$$

The first term on the right is the same thing the other way around so the difference, which is the commutator

$$[b*,\phi] \text{ has Schartz kernel } b(x-y)(\phi(y)-\phi(x)). \tag{2}$$

Now, introducing the cuttoff χ shows that

$$b(x-y)(\phi(y)-\phi(x)) = b(x-y)\chi(x-y)(\phi(y)-\phi(x)) + e(x,y),\ e\in\mathcal{S}(\mathbb{R}^{2n}). \tag{3}$$

The 'error term' $e=(1-\chi(x-y))b(x-y)(\phi(y)-\phi(x))$ certainly contributes a map as in (1) since it corresponds to a map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

The other term in (3) has support in $|x-y|<1$. So consider the Taylor series expansion, with remaineder, of $\phi(y)$ around $y=x$ in this region:

$$\begin{gathered}\phi(y)=\sum_{|\alpha|\le N}\frac{1}{\alpha!}(y-x)^\alpha\phi_\alpha(x)+\sum_{|\alpha|=N+1}(y-x)^\alpha\Phi_\alpha(x,y),\ \phi_\alpha=\partial^\alpha\phi\\ \Phi_\alpha(x,y)=\int_0^1\frac{(1-t)^{N+1}}{N!}\phi_\alpha((1-t)x+ty)dt\\ \Longrightarrow\chi(x-y)\Phi_\alpha(x,y)\in\mathcal{S}(\mathbb{R}^{2n}).\end{gathered} \tag{4}$$

Clearly Φ is infinitely differentiable in all variables in the region $|x-y|<1$ and derivatives are given by similar integrals. Thus to prove the last statement it suffices to observe that on the support of $\chi(x-y)$,

$$\begin{aligned}&\sup_{0\le t\le 1}|\phi_\alpha((1-t)x+ty)|\\ &\qquad\le C_k\inf_{0\le t\le 1,|x-y|<1}(1+|x+t(y-x)|)^{-2k}\le C'_k(1+|x|)^{-k}(1+|y|)^{-k}.\end{aligned} \tag{5}$$

Now to prove (1) for a particular N and M we consider (4) for a possibly much larger N' in place of N. We can take $N'\ge N$ so large that all the terms in the second sum in (4) are the product of a function of $x-y$ of compact support and in $\mathcal{C}^{2p}$, $p>\max(|M|,N+|m|+|M|+1)$ and an element of $\mathcal{S}(\mathbb{R}^{2n})$. Such a kernel clearly defines a map as required for the difference in (1). This means that we get (1) but with N replaced by the possibly larger N' and with

$$\begin{aligned}&b_\alpha(x)=\chi(x)(-x)^\alpha b(x)\Longrightarrow\widehat{b_\alpha}\in S^{m-|\alpha|},\\ &\qquad S^m(\mathbb{R}^n)\ni\hat b\longmapsto\widehat{b_\alpha}\in S^{m-|\alpha|}(\mathbb{R}^n)\text{ linear and continuous.}\end{aligned} \tag{6}$$

However, all the terms with $|\alpha|>N$ (so $|\alpha|\ge N+1$) are just convolutions operators where the Fourier transform gives a symbol of order at most $m-N-1$, followed by multiplication by an function with compact support an element of $\mathcal{S}(\mathbb{R}^n)$. These also gives continuous operators as required of the error term, so the result follows as stated. □

Two results which are essentially corollaries of this are particularly usefull in the low-tech proof of elliptic regularity.

Lemma 1. *For any $m\ge 0$ and any $\phi\in\mathcal{S}(\mathbb{R}^n)$ multiplication by ϕ satisfies*

$$\|\phi u\|_{H^m}\le\sup|\phi|\|u\|_{H^m}+C\|\phi\|_j^{\mathcal{S}}\|u\|_{H^{m-1}} \tag{7}$$

where $\|\phi\|_j^{\mathcal{S}}$ is some continuous seminorm on $\mathcal{S}(\mathbb{R}^n)$ and C and j may depend on m.

Proof. Our standard norm on H^m can be written $\|b_m * u\|_{L^2}$ where

$$\hat{b}_m(\xi) = (1+|\xi|^2)^{m/2} \in S^m(\mathbb{R}^n). \tag{8}$$

Applying (1) for $N = 0$ gives $b_m * \phi u = \phi b_m * u + A$ where $A : H^{m-1}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ and, from the proof above, can be seen to have norm bounded by a seminorm of ϕ. Thus

$$\|\phi u\|_{H^m} = \|b_m * \phi u\|_{L^2} = \|\phi b_m * u + Au\|_{L^2} \le \sup|\phi|\|b_m * u\|_{L^2} + C\|\phi\|_j^{\mathcal{S}}\|u\|_{H^{m-1}} \tag{9}$$

giving (7). □

Now, recall our 'approximate identity' obtained from a fixed function $0 \le \chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with integral 1 by scaling to get

$$\chi_\epsilon(x) = \epsilon^{-n}\chi(\frac{x}{\epsilon}).$$

We showed that if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ then for $\epsilon > 0$, $\chi_\epsilon * u \in \mathcal{C}^\infty(\mathbb{R}^n)$ and if $u \in H^m(\mathbb{R}^n)$ then $\chi_\epsilon * u \to u$ in $H^m(\mathbb{R}^n)$ as $\epsilon \downarrow 0$.

Lemma 2. *With χ_ϵ as above, if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and $Q(x, D)$ is a differential operator of order m with smooth variable coefficients on $\mathbb{R}^n$ then for any s there exists C_s such that*

$$\|\chi_\epsilon * (Q(x, D)u) - Q(x, D)(\chi_\epsilon * u)\|_{H^s} \le C_s\|u\|_{H^{s+m-1}}. \tag{10}$$

Proof. Since u has compact support and $\chi_\epsilon * u$ has only slightly larger support we can always cut off the coefficients of Q outside a compact set, so that they are also in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ without changing the left side of (10). Now, write out the terms $q_\alpha D^\alpha$ and apply (1) with $N = 0$ again, but now $b = D^\alpha v_\epsilon$ and $\phi = q_\alpha$. The important observation is that

$$\widehat{\chi_\epsilon}(\xi) = \hat{\chi}(\epsilon\xi) \in S^0(\mathbb{R}^n) \text{ is bounded in all seminorms.} \tag{11}$$

Once we have this, (10) follows from the last part of the statement of Proposition 1 since this gives a bound on the norm of the error which is independent of ϵ. □

It is these two lemmas which we use in the proof of elliptic regularity. It may seem that the Proposition is overkill, since we only use it for $N = 0$ for the lemmas. However, if you look at the argument, it is rather difficult to get the desired estimate on the error term without 'grinding out' more of the series on the right. In any case Proposition 1 goes a long way to showing the existence of pseudodifferential parametrices – a 'high-tech' proof of elliptic regularity.

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