18.155 LECTURE 15
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Abstract. Notes before and after lecture.

1. Before lecture

- Schwartz' kernel theorem.
- Smoothing operators
- Locality (Peetre's Theorem)
- Proper supports
- Pseudolocality

2. After lecture

- Fixed the confusion between ϕ and ψ in the notes below (I hope).
- I also described the transpose of an operator $L : \mathcal{C}^\infty_c(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega')$ as the operator $L^t : \mathcal{C}^\infty_c(\Omega') \longrightarrow \mathcal{C}^{-\infty}(\Omega)$ with kernel $K^t(x,y) = K(y,x)$ if K is the kernel of L. Notice that it satisfies the identity
$$\langle L\psi, \phi\rangle = \langle \psi, L^t\phi\rangle \tag{1}$$
in terms of the real pairing extending the integral.
- If L and L^t both restrict and extend to define operators $\mathcal{C}^{-\infty}_c(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega')$ and $\mathcal{C}^{-\infty}_c(\Omega') \longrightarrow \mathcal{C}^\infty(\Omega)$ then L (and of course L^t) is called a *smoothing operator*. This is equivalent to the condition $K \in \mathcal{C}^\infty(\Omega' \times \Omega)$.

Having talked about operators on Hilbert space we now want to consider more general operators. The most general type of linear map that arises here is from a 'small' space – a space of test functions – to a 'large' space – a space of distributions. To be concrete let's consider linear maps of two types
$$\begin{gathered} L : \mathcal{C}^\infty_c(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega'),\ \Omega \subset \mathbb{R}^n,\ \Omega' \subset \mathbb{R}^N \text{ open} \\ L : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^N). \end{gathered} \tag{2}$$
In fact we will come to an understanding of the first type through the second.

So consider a linear map of the second type in (2). We need to make some assumptions of continuity. I have not talked very much about the topologies on $\mathcal{S}'(\mathbb{R}^N)$ but let's consider the weakest reasonable one. There are 'obvious' maps on $\mathcal{S}'(\mathbb{R}^N)$ given by pairing with elements of $\mathcal{S}(\mathbb{R}^N)$:
$$e_\phi : \mathcal{S}'(\mathbb{R}^N) \ni u \longrightarrow u(\phi) \in \mathbb{C},\ \phi \in \mathcal{S}(\mathbb{R}^N). \tag{3}$$
There are of course a lot of these maps and we take the weakest topology (fewest open sets) so that these are all continuous. Clearly this requires that
$$e_\phi^{-1}(O) = \{u \in \mathcal{S}'(\mathbb{R}^N) \in O\} \text{ be open for each open set } O \subset \mathbb{C}. \tag{4}$$

So we take these as a sub-basis of the topology – meanin a set in $\mathcal{S}'(\mathbb{R}^N)$ is open if it is an arbitrary union of finite intersections of sets of this type as ϕ and O vary. It is elementary to see that this is a topology and that all the e_ϕ are continuous with respect to it.

In fact we don't really need to worry about these open sets, since we can see that L is continous with respect to the metric topology on the domain and the weak topology on the range space if and only if

$$\mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto (L\psi)(\phi) \in \mathbb{C} \text{ is continuous } \forall\ \phi \in \mathcal{S}(\mathbb{R}^N). \tag{5}$$

I leave it to you to check that this is what continuity means.

Now, how do we get such a linear map? Most of the maps we have been considering, such as $P(D)$ or multiplication by a function of slow growth map say $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$ and certainly have this weak continuity property. An example where $N = n + 1$ for instance is

$$\mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto \delta(x_{n+1})\phi(x_1, \dots, x_n) \in \mathcal{S}'(\mathbb{R}^{n+1}).$$

You can easily check that this satisfies (5).

A much more general linear map comes from the observation that 'exterior tensor product' gives a metrically continuous bilinear map

$$\begin{gathered}\mathcal{S}(\mathbb{R}^N) \times \mathcal{S}(\mathbb{R}^n) \ni (\phi, \psi) \longmapsto \phi(y)\psi(x) = \phi \otimes \psi \in \mathcal{S}(\mathbb{R}^{n+N}), \\ \|\phi \otimes \psi\|_{(k)} \le C\|\psi\|_{(k)}\|\phi\|_{(k)}\end{gathered} \tag{6}$$

for the norms

$$\|\phi\|_{(k)} = \sup_{|\alpha| \le k} (1 + |x|)^k |D^\alpha \phi|. \tag{7}$$

So now, suppose $K \in \mathcal{S}'(\mathbb{R}^{n+N})$ (K for 'kernel') then we can consider the map

$$L : \mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto \left(\mathcal{S}(\mathbb{R}^N) \ni \phi \longmapsto K(\phi \otimes \psi)\right). \tag{8}$$

Why have I written the variables backwards in (6)? Because you are supposed to 'think' that this map is an integral operator

$$(L\psi)(y) = \int K(y, x)\psi(x)dx. \tag{9}$$

Theorem 1 (Schwartz' kernel)**.** *If $K \in \mathcal{S}'(\mathbb{R}^N \times \mathbb{R}^n)$ then the map* (8) *has the continuity property* (5) *and conversely; this gives a linear bijection between kernels and operators.*

Proof. In the forwards direction this is fairly clear. Namely we certainly have

$$|L(\psi)(\phi)| = |K(\phi \otimes \psi)| \le C\|\phi \otimes \psi\|_{(k)} \le C\|\psi\|_{(k)}\|\phi\|_{(k)} \tag{10}$$

The map on the right in (8) is clearly linear for each ψ and by the estimate (10) defines an element of $\mathcal{S}'(\mathbb{R}^N)$ for each ψ. This element depends linearly on ψ so we do get a linear map of the second type in (2). Moreover

$$\mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto e_\phi \circ L(\psi) = K(\phi \otimes \psi) \in \mathbb{C}$$

so (10) also shows that this is continuous in the sense introduced above.

So the challenge is to go the other way, essentially to recover (10) from (5) and then find a kernel. What we have then is the map L and the two conditions (called 'separate continuity')

$$(11)\quad \begin{aligned} &|L(\psi)(\phi)| \le C\|\phi\|_{(k)} \text{ for each } \psi \in \mathcal{S}'(\mathbb{R}^n) \\ &|L(\psi)(\phi)| \le C\|\psi\|_{(j)} \text{ for each } \phi \in \mathcal{S}'(\mathbb{R}^N) \end{aligned}$$

where the constants and the norms can depend on ϕ and ψ respectively. The first is the fact that $L(\psi) \in \mathcal{S}'(\mathbb{R}^N)$ and the first is the continuity (5).

We can apply the uniform boundedness principle here. I did this for Hilbert (maybe Banach) spaces but here we are in the Fréchet world. Nevertheless the same argument works. Look at the sets

$$(12)\quad G_M = \{\phi \in \mathcal{S}(\mathbb{R}^n); |L(\phi)(\psi)| \le M\|\psi\|_{(M)}\|\phi\|_{(M)}\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^N)\},\ M \in \mathbb{N}.$$

These are closed subsets of $\mathcal{S}(\mathbb{R}^n)$ since if $\psi_j \to \psi$ in $\mathcal{S}(\mathbb{R}^n)$ we know that $\|\psi_j\|_{(M)} \to \|\psi\|_{(M)}$ and, by the first part of (11), that $L(\psi_j)(\phi) \to L(\psi)(\phi)$. Similarly by the second part

$$(13)\quad \mathcal{S}(\mathbb{R}^n) = \bigcup_M G_M.$$

So by Baire's Theorem one of these sets has non-empty interior.

Thus for some $\psi' \in \mathcal{S}(\mathbb{R}^n)$ all ψ in a ball of positive radius around 0 satisfy

$$(14)\quad |L(\psi' + \psi)(\psi)| \le M\|\psi + \psi'|_{(M)}\|\phi\|_{(M)}.$$

On this ball $\|\psi + \psi'\|_{(N)}$ is bounded so

$$(15)\quad |L(\psi)(\phi)\| \le |L(\psi' + \psi)(\phi)| + |L(\psi')(\phi)| \le C\|\phi\|_{(N)}$$

for some constant C. However a metric ball contains a ball for one of the norms so, increasing M and C as necessary, we recover the inequality between left and right in (10) for some C, k.

Now, we can replace the norms by weighted Sobolev norms and conclude using duality that

$$(16)\quad L : \langle x\rangle^k H^k(\mathbb{R}^n) \longrightarrow \langle x\rangle^{-k} H^{-k}(\mathbb{R}^N)$$

is continuous for some (larger) k. Next divide and 'integrate' on both sides, replacing L by

$$(17)\quad \tilde{L} = (1 + D^2)^{-M}\langle y\rangle^{-k} L\langle x\rangle^{-k}(1 + D^2)^{-k} : L^2(\mathbb{R}^n) \\ \longrightarrow H^{2M-k}(\mathbb{R}^N) \hookrightarrow \mathcal{C}^0_\infty(\mathbb{R}^N)$$

where we take M large enough to apply the Sobolev embedding theorem to map into the space of bounded continuous functions.

For $\tilde{L}$ we get a kernel by applying the Riesz Representation theorem to the map $L^2(\mathbb{R}^n) \ni \psi \longmapsto (\tilde{L}\psi)(y)$ which gives

$$(18)\quad \tilde{K}(y, x) \in \mathcal{C}^0_\infty(\mathbb{R}^N; L^2(\mathbb{R}^n)) \subset \mathcal{S}'(\mathbb{R}^N \times \mathbb{R}^n)$$

a continuous bounded function with values in L^2. The defining condition means that

$$(19)\quad \tilde{L}(\psi) = \int \tilde{K}(y, x)\psi(x)dx$$

really is an integral operator, although I have been a bit cavalier with complex conjugates.

Finally then the kernel of L is

$$K(y,x) = \langle y\rangle^k (1+D_y^2)^M (1+D^2)^k \langle x\rangle^k \tilde{K}(y,x) \in \mathcal{S}'(\mathbb{R}^N \times \mathbb{R}^n). \tag{20}$$

as follows from (17). □

To handle the more general first case in (2) we can think 'sheaves'. I will leave it to you to check that a distribution

$$K(y,x) \in \mathcal{C}^{-\infty}(\Omega' \times \Omega) \tag{21}$$

does define such a map where our weak continuity condition has become

$$\mathcal{C}^\infty_{\mathrm{c}}(\Omega) \ni \psi \longmapsto L(\psi)(\phi) \in \mathbb{C} \text{ is continuous for each } \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega'). \tag{22}$$

Additionally one can see the uniqueness result that K is determined on $A \times B$ for open subsets if one knows L (defined from K) for all ψ with support in B and ϕ with support in A. This comes from the density of the linear combinations of the products in $\mathcal{C}^\infty_{\mathrm{c}}(A \times B)$. To pass from L to L' of the Schwartz type we have just discussed, insert cut-offs $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$, $\chi = 1$ in a neighbourhood of $\overline{A}$ and similarly for χ'. Then $L' = \chi' L \chi$ is of the Schwartz type in (2) so has a kernel. This recovers L for test functions supported in A and B when these have compact closures, so gives a kernel on $A \times B$. Thus we have a kernel on each of the open sets forming an open cover of $\Omega' \times \Omega$. The uniqueness just alluded to and the sheaf property show this is just one distribution in $\mathcal{C}^{-\infty}(\Omega' \times \Omega)$ which fulfils all the desired conditions.

The Schwartz kernel theorem is not often useful directly, but it does concentrate the mind on the kernels of operators.

What is the kernel of the identity? What is the kernel of a variable coefficient differential operators $P(x, D)$? What does the kernel of our elliptic parameterix $b*$ look like?

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