18.155 LECTURE 11
18 OCTOBER, 2016

RICHARD MELROSE

Abstract. Notes before and after lecture – if you have questions, ask!

Read: Notes Chapter 2.

Unfortunately my proof of the Closed Graph Theorem in lecture was bogus. So as penance I have written out (still fairly brief but now correct I think) proofs of the 'Three Theorems' below.

Before lecture

Hilbert spaces are separable and infinite dimensional unless otherwise stated!

- Theorems: Uniform boundedness, Closed graph, Open mapping.
  I do not plan to prove these, standard proofs rely on Baire's Theorem and can be found in the notes or below.
- Spectrum of an operator. Neumman series.
- Group of invertibles, unitary operators, Kuiper's theorem.
  Again no proof of Kuiper's theorem since we will not use it; see below.
- Finite rank and compact ideals. Calkin algebra.
- Hilbert-Schmidt and trace class ideals. Trace functional.
- Fredholm and semi-Fredholm operators. Density.
- Functional calculus for self-adjoint/normal operators.
- Spectral theorem for compact self-adjoint operators.
- Polar decomposition.

After lecture

- Contrary to the statements above I did go quickly through the proofs of Baire's theorem, the Uniform boundedness principle and the Open Mapping Theorem. I may managed to write a quick outline but not yet.
- I did not talk about unitary operators (yet) nor did I go into Kuiper's Theorem in any detail.
- I showed that $\mathrm{GL}(H) \subset \mathcal{B}(H)$ is open (Neumann series).
- I discussed finite rank operators as a 2-sided $*$ ideal in $\mathcal{B}(H)$.
- The subspace $\mathcal{K}(H) \subset \mathcal{B}(H)$ of compact operators consists of those operators $K \in \mathcal{B}(H)$ such that $K(B(0,1)) \subset M \subset H$ where M is compact.

  Proposition 1. *$\mathcal{K}$ is a closed 2-sided $*$-ideal which is the closure of $\mathcal{R}(H)$, the ideal of finite rank operators.*

  Proof. First we show (I did this in class) that $\overline{\mathcal{R}} \subset \mathcal{K}$. Suppose $\mathcal{R} \ni R_n \longrightarrow K$ in norm. By definition the range of R_n is finite dimensional so give $\epsilon > 0$

there exists $W_n = R_n(H)$ such that

$$\|Ku - R_n u\| < \epsilon/2 \ \forall \ u \in B(0,1). \tag{1}$$

Thus $K(B(0,1))$ is a $\epsilon/2$ close to a finite-dimensional subspace, it is bounded and its closure is both bounded and ϵ close to a finite dimensional subspace. Thus K is compact.

Conversely suppose $K \in \mathcal{K}(H)$. By definition of compactness, applied to $\overline{K(B(0,1))}$, given $\epsilon > 0$ there is a finite dimensional subspace W such that for each $u \in K(B(0,1))$ there exists $w \in W$ such that $\|Ku - w\| < \epsilon$. Let Π_W be the orthogonal projection onto W then $Ku = \Pi_W Ku + (Id - \Pi_W)Ku$ is the orthogonal decomposition of Ku and $\Pi_W Ku \in W$ is the element closest to u, that is

$$\|(\mathrm{Id} - \Pi_W)Ku\| < \epsilon \ \forall \ u \in B(0,1) \Longrightarrow \|K - \Pi_W K\| < \epsilon. \tag{2}$$

This shows that $K \in \overline{\mathcal{R}}$.

By the continuity of products and adjoints it follows that $\mathcal{K}(H)$ is also an ideal and $*$-closed. □

- I mentioned but did not prove that $\mathcal{K}(H)$, $\{0\}$ and $\mathcal{B}(H)$ are the only norm-closed ideals. In fact any other ideal must be contained in $\mathcal{K}$.
- I did not mention explicitly (but will do on Thursday) that

$$K \in \mathcal{K} \Longrightarrow \mathrm{Id} - K \text{ has finite dimensional null space and closed range.} \tag{3}$$

An element in the null space of $\mathrm{Id} - K$ satisfies $u = Ku$, so the unit ball in the null space is its own image under K. It follows that the unit ball of the null space is contained in a compact set and hence is compact. From the homework this week it is therefore finite dimensional.

To see that the range is closed, suppose $f_n = (\mathrm{Id} - K)u_n \longrightarrow f$. We must find $u \in H$ such that $(\mathrm{Id} - K)u = f$. Taking the orthogonal decomposition $u_n = w_n + v_n$ with respect to $\mathrm{Nul}(\mathrm{Id} - K)$ (which is closded) with $v_n \in \mathrm{Nul}(\mathrm{Id} - K)$ it follows that $f_n = (\mathrm{Id} - K)w_n$. We proceed to show that $w_n \to u$ from which it follows that $(\mathrm{Id} - K)u = f$ as claimed. Suppose $\|w_m\|$ was not bounded, then passing to a subsequence we can arrange that $\|w_n\| \to \infty$. Then setting $w_n' = w_n/\|w_n\|$

$$w_n' - Kw_n' = \frac{f_m}{\|w_n\|} \to 0. \tag{4}$$

Since w_n' is bounded, Kw_n' lies in a compact set so has a convergent subsequence and from (4) it follows that, again passing to a subsequence, $w_n' \to w' \in \mathrm{Nul}(\mathrm{Id} - K)^\perp$. Passing to the limit in (4) however $(\mathrm{Id} - Kw) = 0$, so $w = 0$ which contradicts the fact that $\|w\| = 1$. Thus in fact the sequence w_n must be bounded.

Applying the same argument but not to $w_n \in \mathrm{Nul}(\mathrm{Id} - K)^\perp$ it follows that

$$w_n = Kw_n + f_n. \tag{5}$$

Again Kw_n must have a convergent subsequence so w_n must have a convergent subsequence with limit u which satisfies $(\mathrm{Id} - K)u = f$ and we see that the range of $\mathrm{Id} - K$ is closed.

- Thus $\mathrm{Id} - K$, $K \in \mathcal{K}(H)$ is an example of a Fredholm operator.

Definition 1. An element $P \in \mathcal{B}(H)$ is Fredholm if it has finite dimensional null space, closed range and P^* has finite dimensional null space.

Recall that for any bounded operator $\mathrm{Nul}(P^*)$ is the orthocomplement of $\mathrm{Ran}(P)$ whether the latter is closed or not. Certainly $P^*w = 0$ implies that

$$(Pu, w) = (u, P^*w) = 0 \Longrightarrow \mathrm{Nul}(P^*) \perp \mathrm{Ran}(P). \tag{6}$$

The same argument can be reversed, that is if $(Pu, w) = 0$ for all $u \in H$ then $P^*w = 0$ which shows that $\mathrm{Nul}(P^*) = (\mathrm{Ran}(P))^\perp$. Note that $\mathrm{Nul}(P^*)$ is closed but the corresponding orthogonal decomposition is

$$H = \mathrm{Nul}(P^*) \oplus \overline{\mathrm{Ran}(P)} \tag{7}$$

since $\mathrm{Ran}(P)$ need not be closed in general. In particular one cannot drop the explicit statement that $\mathrm{Ran}(P)$ is closed in the definition of a Fredholm operator – you can say $\mathrm{Ran}(P) = (\mathrm{Nul}(P^*)^\perp$ where $\mathrm{Nul}(P^*)$ is finite dimensional.

•

Proposition 2. *An operator is Fredholm if and only if there is a $Q \in \mathcal{B}(H)$ satisfying any one of the following conditions*

(1)
$$QP = \mathrm{Id} - K_1,\ PQ = \mathrm{Id} - K_2,\ K_1, K_2 \in \mathcal{K}(H).$$

(2)
$$QP = \mathrm{Id} - R_1,\ PQ = \mathrm{Id} - R_2,\ R_1, R_2 \in \mathcal{R}(H).$$

(3)
$$QP = \mathrm{Id} - \Pi_{\mathrm{Nul}(P)},\ PQ = \mathrm{Id} - \Pi_{\mathrm{Nul}(P^*)},\ \Pi_{\mathrm{Nul}(P)},\ \Pi_{\mathrm{Nul}(P^*)} \in \mathcal{R}. \tag{8}$$

Proof. The last form, (8) implies the preceding one, which in turn implies the first. Moreover the first form implies that P is Fredholm since from the first identity $\mathrm{Nul}(P) \subset \mathrm{Nul}(QP) \subset \mathrm{Nul}(\mathrm{Id} - K_1)$ is finite-dimensional from the discussion above. From the second identity $\mathrm{Ran}(P) \supset \mathrm{Ran}(PQ) = \mathrm{Ran}(\mathrm{Id} - K_2)$ is, again from the discussion above, closed and of finite codimension – from which it follows that $\mathrm{Ran}(P)$ is closed of finite codimension. So P is Fredholm.

So it suffices to show that if P is Fredholm then there is a bounded operator Q satisfying the (8) (it is in fact unique). We can restrict P to be an operator

$$\tilde{P} : \mathrm{Nul}(P)^\perp \ni u \longmapsto Pu \in \mathrm{Ran}(P) \tag{9}$$

where both domain and range now are Hilbert spaces. Clearly $\tilde{P}$ is a bijection and so by the Open Mapping Theorem has a continuous inverse,

$$\tilde{Q} : \mathrm{Ran}(P) \longrightarrow \mathrm{Nul}(P)^\perp,\ \tilde{Q}\tilde{P} = \tilde{P}\tilde{Q} = \mathrm{Id}\,. \tag{10}$$

Then define $Q : H \longrightarrow H$ to be $\tilde{Q}(\mathrm{Id} - \Pi_{\mathrm{Nul}(P^*)})$ which is bounded and satisfies (8). □

Notice that any operator Q satisfying one of these conditions is also Fredholm.

Here is a brief discussion of the three results on operators arising from completeness, so all based on

Theorem 1 (Baire). *If a non-empty complete metric is written as a countable union of closed subsets*

$$M = \bigcup_n C_n \tag{11}$$

then one of the C_n (at least) must have an interior point.

Proof. We find a contradiction to the assumption that none of the C_n has an interior point. Start with $C_1 \neq M$ since otherwise it has an interior point so we can find

$$x_1 \in M \setminus C,\ \epsilon_1 > 0 \text{ s.t. } B(x_1, \epsilon_1) \cap C_1 = \emptyset$$

where we use the assumption that C_1 is closed, so its complement is open. Next, $B(x_1, \frac{1}{3}\epsilon_1)$ cannot be contained in C_2 so there must exist

$$x_2 \in B(x_1, \frac{1}{3}\epsilon_1),\ \epsilon_2 > 0,\ \epsilon_2 < \frac{1}{3}\epsilon_1,\ B(x_2, \epsilon_2) \cap C_2 = \emptyset. \tag{12}$$

The conditions ensure that $B(x_2, \epsilon_2) \subset B(x_1, \frac{2}{3}\epsilon_1)$ so the smaller ball is disjoint from C_1 as well. Now proceed by induction and so construct a sequence where for all $j \geq 2$,

$$x_j \in B(x_{j-1}, \frac{1}{3}\epsilon_{j-1}),\ \epsilon_j > 0,\ \epsilon_j < \frac{1}{3}\epsilon_{j-1},\ B(x_j, \epsilon_j) \cap C_j = \emptyset.$$

It follows that $\epsilon_j < 3^{1-j}\epsilon_1$ so is summable and hence $\{x_j\}$ is Cauchy so converges by the assumed completeness. The limit $x \in \overline{B(x_n, \epsilon_n)}$ for all n, since the sequence is eventually in the open ball. The conditions above show that $x \notin C_n$ for all n which is the contradiction. $\square$

Now with this we can prove

Theorem 2 (Uniform Boundedness=Banach-Steinhaus). *Suppose B and N are respectively a Banach and a normed space and $\mathcal{L} \subset \mathcal{B}(B, N)$ is a subset of the bounded operators which is 'pointwise bounded' in the sense that*

$$\text{for each } u \in B,\ \{Lu; L \in \mathcal{L}\} \subset N \text{ is bounded} \tag{13}$$

then $\mathcal{L}$ is bounded (in norm).

The fact that N need not be complete is rather bogus since we can always replace it by its completion and nothing changes.

Proof. For each n set

$$C_n = \{u \in B; \|u\| \leq 1,\ \|Lu\| \leq n\ \forall\ L \in \mathcal{L}\}. \tag{14}$$

The assumption (13) shows that

$$\{u \in B; \|u\| \leq 1\} = \bigcup_n C_n$$

and each C_n is closed by the assumtion that each $L \in \mathcal{L}$ is bounded, i.e. continuous. So Baire's Theorem applies and this means there exists $\epsilon > 0$ and u such that

$$v \in B,\ \|v\| < \epsilon \Longrightarrow \|L(u+v)\| \leq n \Longrightarrow \|Lv\| \leq n + \|Lu\| \leq 2n,\ \forall\ L \in \mathcal{L} \tag{15}$$

So in fact $\|L\| \leq 2n/\epsilon$ for all $L \in \mathcal{L}$. $\square$

Next in order now is

Theorem 3 (Open Mapping)**.** *If $L : B_1 \longrightarrow B_2$ is a bounded surjective map between Banach spaces then it is* open

$$L(O) \subset B_2 \text{ is open for each open set } O \subset B_1. \tag{16}$$

Proof. Start with the most important case that L is actually a bijection. Then we try to show that the inverse image of the ball of radius one in B_2 has an interior point in B_1 by setting

$$E_N = L(\{u \in B_1; \|u\| < N,\ \|Lu\| < 1\}) \subset \{f \in B_2; \|f\| < 1\} = \bigcup E_N \tag{17}$$

by surjectivity. The problem is that we do not know that the E_N are closed (this is the same problem that dooms the proof of the Closed Graph Theorem that I tried to give in class, that can presumably be corrected in the same way). So, just let C_N be the closure of E_N and Baire's Theorem does apply. So for some N there is a ball of positive radius contained in the *closure* of the image of $\{u \in B_1; \|u\| < N\}$ under L. By surjectivity the centre is the image of some point so subtracting that and scaling using the linearity of L what we conclude is that for some p

$$f \in B_2,\ \|f\| \le 1 \Longrightarrow \exists\ u_n \in B_1,\ \|u_n\| \le p,\ Lu_n \to f. \tag{18}$$

The problem of course is that we do not immediately know that $u_n \to u$.

To see this back off a little from (18) and just use the fact that we can get arbitrarily close with the sequence, then scale again to to see that

$$f \in B_2 \Longrightarrow\ \exists\ v \in B_1,\ \|v\| \le p\|f\| \text{ s.t. } \|f - Lv\| \le \frac{1}{2}\|f\|. \tag{19}$$

Choose such a $v = v_1$ and then iteratively choose a sequence $f_n \in B_2$ $v_n \in B_1$, where $f_0 = f$ and

$$\begin{aligned} f_n = f_{n-1} - Lv_{n-1},\ \|f_n - Lv_n\| \le \frac{1}{2}\|f_n\|,\ \|v\|_n \le p\|f_n\| \\ \Longrightarrow \|f_{n+1}\| \le \frac{1}{2}\|f_n\|,\ n \ge 1. \end{aligned}$$

So in fact $\|f_n\| \le 2^{-n}\|f\|$ and hence $\|v_n\| \le 2^{-n}p\|f\|$ so the series v_n is summable (Cauchy in a complete space) and

$$u = \sum_n v_n \text{ satisfies } Lu = \sum_n Lv_n = \sum_n (f_{n-1} - f_n) = f,\ \|u\| \le 2p\|f\|. \tag{20}$$

Since L is bijective this solution is the unique one and this proves that the inverse is bounded and hence the map is open.

For the general case of a surjective linear map in the case of Hilbert spaces as domain apply this special result to $\tilde{L} : \mathrm{Nul}(L)^\perp \longrightarrow B_2$ which is therefore open and check by hand that the projection map $\Pi : H \longrightarrow \mathrm{Nul}(L)^\perp$ is open. For the general case of a Banach space as domain use quotients instead. □

Theorem 4 (Closed Graph)**.** *A linear map between Banach spaces $L : B_1 \longrightarrow B_2$ is bounded if and only if its graph*

$$\mathrm{Gr}(L) = \{(u, Lu); u \in B_1\} \subset B_1 \times B_2 \tag{21}$$

is closed.

Proof. If L is bounded, i.e. continuous, then $(u_n, Lu_n) \to (u, f)$ in $B_1 \times B_2$ implies $u_n \to u$ and hence $Lu_n \to Lu = f$ so the limit is in $\mathrm{Gr}(L)$ which is therefore closed.

Conversely suppose the graph is closed, it is then a Banach space with the norm $\|u\|_{B_1} + \|Lu\|_{B_2}$. The projection operators π_1 and π_2 from $B_1 \times B_2$ (which is a Banach space) to B_1 and B_2 are both continuous. By definition of a map the restriction

$$\pi_1' : \mathrm{Gr}(L) \longrightarrow B_1 \tag{22}$$

is a bijection and bounded. So when $\mathrm{Gr}(L)$ is closed we can use the Open Mapping Theorem to see that $(\pi_1')^{-1} : B_1 \longrightarrow \mathrm{Gr}(L)$ is also bounded. However $L = \pi_2 \circ (\pi_1')^{-1}$ so it is also bounded. $\square$

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: `rbm@math.mit.edu`