CHAPTER 3

Distributions

1. Test functions

So far we have largely been dealing with integration. One thing we have seen is that, by considering dual spaces, we can think of functions as functionals. Let me briefly review this idea.

Consider the unit ball in $\mathbb{R}^n$,

$$\overline{\mathbb{B}}^n = \{x \in \mathbb{R}^n \,;\, |x| \le 1\} \;.$$

I take the *closed* unit ball because I want to deal with a compact metric space. We have dealt with several Banach spaces of functions on $\overline{\mathbb{B}^n}$, for example

$$C(\overline{\mathbb{B}^n}) = \left\{u : \overline{\mathbb{B}^n} \to \mathbb{C} \,;\, u \text{ continuous}\right\}$$
$$L^2(\overline{\mathbb{B}^n}) = \left\{u : \overline{\mathbb{B}^n} \to \mathbb{C}; \text{Borel measurable with } \int |u|^2 \, dx < \infty\right\}.$$

Here, as always below, dx is Lebesgue measure and functions are identified if they are equal almost everywhere.

Since $\overline{\mathbb{B}^n}$ is compact we have a natural inclusion

$$C(\overline{\mathbb{B}^n}) \hookrightarrow L^2(\overline{\mathbb{B}^n})\,. \tag{1.1}$$

This is also a topological inclusion, i.e., is a bounded linear map, since

$$\|u\|_{L^2} \le C\|u\|_\infty \tag{1.2}$$

where C^2 is the volume of the unit ball.

In general if we have such a set up then

Lemma 1.1. *If $V \hookrightarrow U$ is a subspace with a stronger norm,*

$$\|\varphi\|_U \le C\|\varphi\|_V \;\forall\; \varphi \in V$$

then restriction gives a continuous linear map

$$U' \to V',\; U' \ni L \longmapsto \tilde{L} = L|_V \in V',\; \|\tilde{L}\|_{V'} \le C\|L\|_{U'}\,. \tag{1.3}$$

If V is dense in U then the map (6.9) *is injective.*

PROOF. By definition of the dual norm

$$\begin{aligned}\|\tilde{L}\|_{V'} &= \sup\left\{\left|\tilde{L}(v)\right| \; ; \; \|v\|_V \le 1\,,\; v \in V\right\} \\ &\le \sup\left\{\left|\tilde{L}(v)\right| \; ; \; \|v\|_U \le C\,,\; v \in V\right\} \\ &\le \sup\left\{|L(u)| \; ; \; \|u\|_U \le C\,,\; u \in U\right\} \\ &= C\|L\|_{U'}\,.\end{aligned}$$

If $V \subset U$ is dense then the vanishing of $L : U \to \mathbb{C}$ on V implies its vanishing on U.

□

Going back to the particular case (6.8) we do indeed get a continuous map between the dual spaces

$$L^2(\overline{\mathbb{B}^n}) \cong (L^2(\overline{\mathbb{B}^n}))' \to (C(\overline{\mathbb{B}^n}))' = M(\overline{\mathbb{B}^n})\,.$$

Here we use the Riesz representation theorem and duality for Hilbert spaces. The map use here is supposed to be *linear* not antilinear, i.e.,

$$L^2(\overline{\mathbb{B}^n}) \ni g \longmapsto \int \cdot g\, dx \in (C(\overline{\mathbb{B}^n}))'\,. \tag{1.4}$$

So the idea is to make the space of 'test functions' as small as reasonably possible, while still retaining *density* in reasonable spaces.

Recall that a function $u : \mathbb{R}^n \to \mathbb{C}$ is *differentiable* at $\overline{x} \in \mathbb{R}^n$ if there exists $a \in \mathbb{C}^n$ such that

$$|u(x) - u(\overline{x}) - a \cdot (x - \overline{x})| = o(|x - \overline{x}|)\,. \tag{1.5}$$

The 'little oh' notation here means that given $\epsilon > 0$ there exists $\delta > 0$ s.t.

$$|x - \overline{x}| < \delta \Rightarrow |u(x) - u(\overline{x}) - a(x - \overline{x})| < \epsilon\,|x - \overline{x}|\,.$$

The coefficients of $a = (a_1, \dots, a_n)$ are the partial derivations of u at $\overline{x}$,

$$a_i = \frac{\partial u}{\partial x_j}(\overline{x})$$

since

$$a_i = \lim_{t \to 0} \frac{u(\overline{x} + te_i) - u(\overline{x})}{t}\,, \tag{1.6}$$

$e_i = (0, \dots, 1, 0, \dots, 0)$ being the ith basis vector. The function u is said to be *continuously differentiable* on $\mathbb{R}^n$ if it is differentiable at *each* point $\overline{x} \in \mathbb{R}^n$ and each of the n partial derivatives are continuous,

$$\frac{\partial u}{\partial x_j} : \mathbb{R}^n \to \mathbb{C}\,. \tag{1.7}$$

DEFINITION 1.2. *Let $\mathcal{C}_0^1(\mathbb{R}^n)$ be the subspace of $\mathcal{C}_0(\mathbb{R}^n) = \mathcal{C}_0^0(\mathbb{R}^n)$ such that each element $u \in \mathcal{C}_0^1(\mathbb{R}^n)$ is continuously differentiable and $\frac{\partial u}{\partial x_j} \in \mathcal{C}_0(\mathbb{R}^n)$, $j = 1, \dots, n$.*

PROPOSITION 1.3. *The function*

$$\|u\|_{\mathcal{C}^1} = \|u\|_\infty + \sum_{i=1}^n \|\frac{\partial u}{\partial x_1}\|_\infty$$

is a norm on $\mathcal{C}_0^1(\mathbb{R}^n)$ with respect to which it is a Banach space.

PROOF. That $\| \ \|_{\mathcal{C}^1}$ is a norm follows from the properties of $\| \ \|_\infty$. Namely $\|u\|_{\mathcal{C}^1} = 0$ certainly implies $u = 0$, $\|au\|_{\mathcal{C}^1} = |a| \ \|u\|_{\mathcal{C}^1}$ and the triangle inequality follows from the same inequality for $\| \ \|_\infty$.

Similarly, the main part of the completeness of $\mathcal{C}_0^1(\mathbb{R}^n)$ follows from the completeness of $\mathcal{C}_0^0(\mathbb{R}^n)$. If $\{u_n\}$ is a Cauchy sequence in $\mathcal{C}_0^1(\mathbb{R}^n)$ then u_n and the $\frac{\partial u_n}{\partial x_j}$ are Cauchy in $\mathcal{C}_0^0(\mathbb{R}^n)$. It follows that there are limits of these sequences,

$$u_n \to v \, , \ \frac{\partial u_n}{\partial x_j} \to v_j \in \mathcal{C}_0^0(\mathbb{R}^n) \, .$$

However we do have to check that v is continuously differentiable and that $\frac{\partial v}{\partial x_j} = v_j$.

One way to do this is to use the Fundamental Theorem of Calculus in each variable. Thus

$$u_n(\overline{x} + te_i) = \int_0^t \frac{\partial u_n}{\partial x_j}(\overline{x} + se_i) \, ds + u_n(\overline{x}) \, .$$

As $n \to \infty$ all terms converge and so, by the continuity of the integral,

$$u(\overline{x} + te_i) = \int_0^t v_j(\overline{x} + se_i) \, ds + u(\overline{x}) \, .$$

This shows that the limit in (6.20) exists, so $v_i(\overline{x})$ is the partial derivation of u with respect to x_i. It remains only to show that u is indeed differentiable at each point and I leave this to you in Problem 17.

□

So, almost by definition, we have an example of Lemma 6.17,

$$\mathcal{C}_0^1(\mathbb{R}^n) \hookrightarrow \mathcal{C}_0^0(\mathbb{R}^n).$$

It is in fact dense but I will not bother showing this (yet). So we know that

$$(\mathcal{C}_0^0(\mathbb{R}^n))' \to (\mathcal{C}_0^1(\mathbb{R}^n))'$$

and we expect it to be injective. Thus there are *more* functionals on $\mathcal{C}_0^1(\mathbb{R}^n)$ including things that are ‘more singular than measures’.

An example is related to the Dirac delta

$$\delta(\overline{x})(u) = u(\overline{x})\,,\ u \in \mathcal{C}_0^0(\mathbb{R}^n)\,,$$

namely

$$\mathcal{C}_0^1(\mathbb{R}^n) \ni u \longmapsto \frac{\partial u}{\partial x_j}(\overline{x}) \in \mathbb{C}\,.$$

This is clearly a continuous linear functional which it is only just to denote $\frac{\partial}{\partial x_j}\delta(\overline{x})$.

Of course, why stop at one derivative?

Definition 1.4. *The space* $\mathcal{C}_0^k(\mathbb{R}^n) \subset \mathcal{C}_0^1(\mathbb{R}^n)\, k \geq 1$ *is defined inductively by requiring that*

$$\frac{\partial u}{\partial x_j} \in \mathcal{C}_0^{k-1}(\mathbb{R}^n)\,,\ j = 1, \dots, n\,.$$

The norm on $\mathcal{C}_0^k(\mathbb{R}^n)$ *is taken to be*

$$\|u\|_{\mathcal{C}^k} = \|u\|_{\mathcal{C}^{k-1}} + \sum_{j=1}^{n} \|\frac{\partial u}{\partial x_j}\|_{\mathcal{C}^{k-1}}\,. \tag{1.8}$$

These are all Banach spaces, since if $\{u_n\}$ is Cauchy in $\mathcal{C}_0^k(\mathbb{R}^n)$, it is Cauchy and hence convergent in $\mathcal{C}_0^{k-1}(\mathbb{R}^n)$, as is $\partial u_n/\partial x_j$, $j = 1, \dots, n-1$. Furthermore the limits of the $\partial u_n/\partial x_j$ are the derivatives of the limits by Proposition 1.3.

This gives us a sequence of spaces getting 'smoother and smoother'

$$\mathcal{C}_0^0(\mathbb{R}^n) \supset \mathcal{C}_0^1(\mathbb{R}^n) \supset \cdots \supset \mathcal{C}_0^k(\mathbb{R}^n) \supset \cdots\,,$$

with norms getting larger and larger. The duals can also be expected to get larger and larger as k increases.

As well as looking at functions getting smoother and smoother, we need to think about 'infinity', since $\mathbb{R}^n$ is not compact. Observe that an element $g \in L^1(\mathbb{R}^n)$ (with respect to Lebesgue measure by default) defines a functional on $\mathcal{C}_0^0(\mathbb{R}^n)$ — and hence *all* the $\mathcal{C}_0^k(\mathbb{R}^n)$s. However a function such as the constant function 1 is *not* integrable on $\mathbb{R}^n$. Since we certainly want to talk about this, and polynomials, we consider a second condition of *smallness at infinity.* Let us set

$$\langle x\rangle = (1 + |x|^2)^{1/2} \tag{1.9}$$

a function which is the size of $|x|$ for $|x|$ large, but has the virtue of being smooth[1]

[1] See Problem 18.

DEFINITION 1.5. *For any $k,\ l \in \mathbb{N} = \{1, 2, \cdots\}$ set*

$$\langle x\rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n) = \left\{u \in \mathcal{C}_0^k(\mathbb{R}^n)\,;\, u = \langle x\rangle^{-l}v\,,\ v \in \mathcal{C}_0^k(\mathbb{R}^n)\right\}\,,$$

with norm, $\|u\|_{k,l} = \|v\|_{\mathcal{C}^k}\,,\ v = \langle x\rangle^l u$.

Notice that the definition just says that $u = \langle x\rangle^{-l}v$, with $v \in \mathcal{C}_0^k(\mathbb{R}^n)$. It follows immediately that $\langle x\rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n)$ is a Banach space with this norm.

DEFINITION 1.6. *Schwartz' space*[2] *of test functions on $\mathbb{R}^n$ is*

$$\mathcal{S}(\mathbb{R}^n) = \left\{u : \mathbb{R}^n \to \mathbb{C}; u \in \langle x\rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n) \text{ for all } k \text{ and } l \in \mathbb{N}\right\}.$$

It is not immediately apparent that this space is non-empty (well 0 is in there but...); that

$$P(x)\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n) \tag{1.10}$$

for any polynomial P is Problem 19.

COROLLARY 1.7. *$\mathcal{S}(\mathbb{R}^n)$ is infinite-dimensional.*

In fact the linear space in (1.10) turns out to be dense in $\mathcal{S}(\mathbb{R}^n)$ when we sort out the topology – so it will be separable.

Schwartz' idea is that the dual of $\mathcal{S}(\mathbb{R}^n)$ should contain all the 'interesting' objects, at least those of 'polynomial growth'. The problem is that we do *not* have a good norm on $\mathcal{S}(\mathbb{R}^n)$. Rather we have a *lot* of them. Observe that

$$\langle x\rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n) \subset \langle x\rangle^{-l'}\mathcal{C}_0^{k'}(\mathbb{R}^n) \text{ if } l \ge l' \text{ and } k \ge k'\,.$$

Thus we see that as a linear space

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_k \langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n). \tag{1.11}$$

Since these spaces are getting smaller, we have a countably infinite number of norms. For this reason $\mathcal{S}(\mathbb{R}^n)$ is called a *countably normed* space.

PROPOSITION 1.8. *For $u \in \mathcal{S}(\mathbb{R}^n)$, set*

$$\|u\|_{(k)} = \|\langle x\rangle^k u\|_{\mathcal{C}^k} \tag{1.12}$$

and define

$$d(u, v) = \sum_{k=0}^{\infty} 2^{-k}\frac{\|u - v\|_{(k)}}{1 + \|u - v\|_{(k)}}, \tag{1.13}$$

then d is a distance function in $\mathcal{S}(\mathbb{R}^n)$ with respect to which it is a complete metric space.

[2]Laurent Schwartz – this one with a 't'.

PROOF. The series in (1.13) certainly converges, since

$$\frac{\|u-v\|_{(k)}}{1+\|u-v\|_{(k)}} \le 1.$$

The first two conditions on a metric are clear,

$$d(u,v) = 0 \Rightarrow \|u-v\|_{\mathcal{C}_0} = 0 \Rightarrow u = v,$$

and symmetry is immediate. The triangle inequality is perhaps more mysterious!

Certainly it is enough to show that

$$\tilde{d}(u,v) = \frac{\|u-v\|}{1+\|u-v\|} \tag{1.14}$$

is a metric on any normed space, since then we may sum over k. Thus we consider

$$\begin{aligned}\frac{\|u-v\|}{1+\|u-v\|} + \frac{\|v-w\|}{1+\|v-w\|}\\ = \frac{\|u-v\|(1+\|v-w\|)+\|v-w\|(1+\|u-v\|)}{(1+\|u-v\|)(1+\|v-w\|)}.\end{aligned}$$

Comparing this to $\tilde{d}(v,w)$ we must show that

$$\begin{aligned}&(1+\|u-v\|)(1+\|v-w\|)\|u-w\|\\ &\quad\le (\|u-v\|(1+\|v-w\|)+\|v-w\|(1+\|u-v\|))(1+\|u-w\|).\end{aligned}$$

Starting from the LHS and using the triangle inequality,

$$\begin{aligned}\text{LHS } &\le \|u-w\| + (\|u-v\|+\|v-w\|+\|u-v\|\|v-w\|)\|u-w\|\\ &\le (\|u-v\|+\|v-w\|+\|u-v\|\|v-w\|)(1+\|u-w\|)\\ &\le \text{RHS}.\end{aligned}$$

Thus, d is a metric.

Suppose u_n is a Cauchy sequence. Thus, $d(u_n,u_m) \to 0$ as $n,m \to \infty$. In particular, given

$$\begin{aligned}&\epsilon > 0\ \exists\ N \text{ s.t. } n,m > N \text{ implies}\\ &\quad d(u_n,u_m) < \epsilon 2^{-k}\ \forall\ n,m > N.\end{aligned}$$

The terms in (1.13) are all positive, so this implies

$$\frac{\|u_n-u_m\|_{(k)}}{1+\|u_n-u_m\|_{(k)}} < \epsilon\ \forall\ n,m > N.$$

If $\epsilon < 1/2$ this in turn implies that

$$\|u_n-u_m\|_{(k)} < 2\epsilon,$$

so the sequence is Cauchy in $\langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n)$ for each k. From the completeness of these spaces it follows that $u_n \to u$ in $\langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n)_j$ for each k. Given $\epsilon > 0$ choose k so large that $2^{-k} < \epsilon/2$. Then $\exists$ N s.t. $n > N$

$$\Rightarrow \|u - u_n\|_{(j)} < \epsilon/2 \ n > N, \ j \leq k.$$

Hence

$$\begin{aligned} d(u_n, u) &= \sum_{j\leq k} 2^{-j} \frac{\|u - u_n\|_{(j)}}{1 + \|u - u_n\|_{(j)}} \\ &+ \sum_{j>k} 2^{-j} \frac{\|u - u_n\|_{(j)}}{1 + \|u - u_n\|_{(j)}} \\ &\leq \epsilon/4 + 2^{-k} < \epsilon. \end{aligned}$$

This $u_n \to u$ in $\mathcal{S}(\mathbb{R}^n)$. $\square$

As well as the Schwartz space, $\mathcal{S}(\mathbb{R}^n)$, of functions of rapid decrease with all derivatives, there is a smaller 'standard' space of test functions, namely

$$\mathcal{C}_c^\infty(\mathbb{R}^n) = \{u \in \mathcal{S}(\mathbb{R}^n); \operatorname{supp}(u) \Subset \mathbb{R}^n\}, \tag{1.15}$$

the space of smooth functions of compact support. Again, it is not quite obvious that this has any non-trivial elements, but it does as we shall see. If we fix a compact subset of $\mathbb{R}^n$ and look at functions with support in that set, for instance the closed ball of radius $R > 0$, then we get a closed subspace of $\mathcal{S}(\mathbb{R}^n)$, hence a complete metric space. One 'problem' with $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is that it does not have a complete metric topology which restricts to this topology on the subsets. Rather we must use an *inductive limit* procedure to get a decent topology.

Just to show that this is not really hard, I will discuss it briefly here, but it is not used in the sequel. In particular I will not do this in the lectures themselves. By definition our space $\mathcal{C}_c^\infty(\mathbb{R}^n)$ (denoted traditionally as $\mathcal{D}(\mathbb{R}^n)$) is a countable union of subspaces

$$\mathcal{C}_c^\infty(\mathbb{R}^n) = \bigcup_{n\in\mathbb{N}} \dot{\mathcal{C}}_c^\infty(B(n)), \ \dot{\mathcal{C}}_c^\infty(B(n)) = \{u \in \mathcal{S}(\mathbb{R}^n); u = 0 \text{ in } |x| > n\}. \tag{1.16}$$

Consider

$$\mathcal{T} = \{U \subset \mathcal{C}_c^\infty(\mathbb{R}^n); U \cap \dot{\mathcal{C}}_c^\infty(B(n)) \text{ is open in } \dot{\mathcal{C}}^\infty(B(n)) \text{ for each } n\}. \tag{1.17}$$

This is a topology on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ – contains the empty set and the whole space and is closed under finite intersections and arbitrary unions –

simply because the same is true for the open sets in $\dot{\mathcal{C}}^\infty(B(n))$ for each n. This is in fact the inductive limit topology. One obvious question is:- what does it mean for a linear functional $u : \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathbb{C}$ to be continuous? This just means that $u^{-1}(O)$ is open for each open set in $\mathbb{C}$. Directly from the definition this in turn means that $u^{-1}(O) \cap \dot{\mathcal{C}}^\infty(B(n))$ should be open in $\dot{\mathcal{C}}^\infty(B(n))$ for each n. This however just means that, restricted to each of these subspaces u is continuous. If you now go forwards to Lemma 2.3 you can see what this means; see Problem 74.

Of course there is a lot more to be said about these spaces; you can find plenty of it in the references.

2. Tempered distributions

A good first reference for distributions is [**2**], [**5**] gives a more exhaustive treatment.

The complete metric topology on $\mathcal{S}(\mathbb{R}^n)$ is described above. Next I want to try to convice you that elements of its dual space $\mathcal{S}'(\mathbb{R}^n)$, have enough of the properties of functions that we can work with them as 'generalized functions'.

First let me develop some notation. A differentiable function $\varphi : \mathbb{R}^n \to \mathbb{C}$ has partial derivatives which we have denoted $\partial\varphi/\partial x_j : \mathbb{R}^n \to \mathbb{C}$. For reasons that will become clear later, we put a $\sqrt{-1}$ into the definition and write

$$D_j \varphi = \frac{1}{i} \frac{\partial \varphi}{\partial x_j}\,. \tag{2.1}$$

We say φ is once continuously differentiable if each of these $D_j\varphi$ is continuous. Then we defined k times continuous differentiability inductively by saying that φ and the $D_j\varphi$ are $(k-1)$-times continuously differentiable. For $k = 2$ this means that

$$D_j D_k \varphi \text{ are continuous for } j, k = 1, \cdots, n\,.$$

Now, recall that, if continuous, these second derivatives are symmetric:

$$D_j D_k \varphi = D_k D_j \varphi\,. \tag{2.2}$$

This means we can use a compact notation for higher derivatives. Put $\mathbb{N}_0 = \{0, 1, \ldots\}$; we call an element $\alpha \in \mathbb{N}_0^n$ a 'multi-index' and if φ is at least k times continuously differentiable, we set[3]

$$D^\alpha \varphi = \frac{1}{i^{|\alpha|}} \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n} \varphi \text{ whenever } |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \le k. \tag{2.3}$$

[3]Periodically there is the possibility of confusion between the two meanings of $|\alpha|$ but it seldom arises.

In fact we will use a closely related notation of powers of a variable. Namely if α is a multi-index we shall also write

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}. \tag{2.4}$$

Now we have *defined* the spaces.

$$\mathcal{C}_0^k(\mathbb{R}^n) = \left\{\varphi : \mathbb{R}^n \to \mathbb{C}\,;\, D^\alpha\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| \le k\right\}\,. \tag{2.5}$$

Notice the convention is that $D^\alpha\varphi$ is asserted to exist if it is required to be continuous! Using $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ we defined

$$\langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n) = \left\{\varphi : \mathbb{R}^n \to \mathbb{C}\,;\, \langle x\rangle^k\varphi \in \mathcal{C}_0^k(\mathbb{R}^n)\right\}\,, \tag{2.6}$$

and then our space of test functions is

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_k \langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n)\,.$$

Thus,

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \Leftrightarrow D^\alpha(\langle x\rangle^k\varphi) \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| \le k \text{ and all } k\,. \tag{2.7}$$

Lemma 2.1. *The condition $\varphi \in \mathcal{S}(\mathbb{R}^n)$ can be written*

$$\langle x\rangle^k D^\alpha\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| \le k\,,\ \forall\ k\,.$$

Proof. We first check that

$$\begin{aligned}&\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\,,\ D_j(\langle x\rangle\varphi) \in \mathcal{C}_0^0(\mathbb{R}^n)\,,\ j = 1,\cdots,n\\ \Leftrightarrow\ &\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\,,\ \langle x\rangle D_j\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\,,\ j = 1,\cdots,n\,.\end{aligned}$$

Since

$$D_j\langle x\rangle\varphi = \langle x\rangle D_j\varphi + (D_j\langle x\rangle)\varphi$$

and $D_j\langle x\rangle = \frac{1}{i}x_j\langle x\rangle^{-1}$ is a bounded continuous function, this is clear. Then consider the same thing for a larger k:

$$\begin{aligned}&D^\alpha\langle x\rangle^p\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| = p\,,\ 0 \le p \le k\\ \Leftrightarrow\ &\langle x\rangle^p D^\alpha\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| = p\,,\ 0 \le p \le k\,.\end{aligned} \tag{2.8}$$

□

I leave you to check this as Problem 2.1.

Corollary 2.2. *For any $k \in \mathbb{N}$ the norms*

$$\|\langle x\rangle^k\varphi\|_{\mathcal{C}^k} \text{ and } \sum_{\substack{|\alpha|\le k,\\ |\beta|\le k}} \|x^\alpha D_x^\beta\varphi\|_\infty$$

are equivalent.

PROOF. Any reasonable proof of (2.2) shows that the norms

$$\|\langle x\rangle^k\varphi\|_{\mathcal{C}^k} \text{ and } \sum_{|\beta|\le k}\|\langle x\rangle^k D^\beta\varphi\|_\infty$$

are equivalent. Since there are positive constants such that

$$C_1\left(1+\sum_{|\alpha|\le k}|x^\alpha|\right)\le\langle x\rangle^k\le C_2\left(1+\sum_{|\alpha|\le k}|x^\alpha|\right)$$

the equivalent of the norms follows.

□

PROPOSITION 2.3. *A linear functional* $u:\mathcal{S}(\mathbb{R}^n)\to\mathbb{C}$ *is continuous if and only if there exist* C, k *such that*

$$|u(\varphi)|\le C\sum_{\substack{|\alpha|\le k,\\ |\beta|\le k}}\sup_{\mathbb{R}^n}\left|x^\alpha D_x^\beta\varphi\right|.$$

PROOF. This is just the equivalence of the norms, since we showed that $u\in\mathcal{S}'(\mathbb{R}^n)$ if and only if

$$|u(\varphi)|\le C\|\langle x\rangle^k\varphi\|_{\mathcal{C}^k}$$

for some k.

□

LEMMA 2.4. *A linear* map

$$T:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$$

is continuous if and only if for each k *there exist* C *and* j *such that if* $|\alpha|\le k$ *and* $|\beta|\le k$

$$\sup\left|x^\alpha D^\beta T\varphi\right|\le C\sum_{|\alpha'|\le j,\ |\beta'|\le j}\sup_{\mathbb{R}^n}\left|x^{\alpha'}D^{\beta'}\varphi\right|\ \forall\ \varphi\in\mathcal{S}(\mathbb{R}^n). \tag{2.9}$$

PROOF. This is Problem 2.2. □

All this messing about with norms shows that

$$x_j:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n) \text{ and } D_j:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$$

are continuous.

So now we have some idea of what $u\in\mathcal{S}'(\mathbb{R}^n)$ means. Let's notice that $u\in\mathcal{S}'(\mathbb{R}^n)$ implies

$$x_ju\in\mathcal{S}'(\mathbb{R}^n)\ \forall\ j=1,\cdots,n \tag{2.10}$$

$$D_ju\in\mathcal{S}'(\mathbb{R}^n)\ \forall\ j=1,\cdots,n \tag{2.11}$$

$$\varphi u\in\mathcal{S}'(\mathbb{R}^n)\ \forall\ \varphi\in\mathcal{S}(\mathbb{R}^n) \tag{2.12}$$

where we have to *define* these things in a reasonable way. Remember that $u \in \mathcal{S}'(\mathbb{R}^n)$ is "supposed" to be like an integral against a "generalized function"

$$u(\psi) = \int_{\mathbb{R}^n} u(x)\psi(x)\,dx \;\forall\; \psi \in \mathcal{S}(\mathbb{R}^n). \tag{2.13}$$

Since it would be true if u *were* a function we *define*

$$x_j u(\psi) = u(x_j\psi) \;\forall\; \psi \in \mathcal{S}(\mathbb{R}^n). \tag{2.14}$$

Then we check that $x_j u \in \mathcal{S}'(\mathbb{R}^n)$:

$$\begin{aligned} |x_j u(\psi)| &= |u(x_j\psi)| \\ &\le C \sum_{|\alpha|\le k,\ |\beta|\le k} \sup_{\mathbb{R}^n} \left| x^\alpha D^\beta(x_j\psi)\right| \\ &\le C' \sum_{|\alpha|\le k+1,\ |\beta|\le k} \sup_{\mathbb{R}^n} \left| x^\alpha D^\beta\psi\right| \,. \end{aligned}$$

Similarly we can define the partial *derivatives* by using the standard integration by parts formula

$$\int_{\mathbb{R}^n} (D_j u)(x)\varphi(x)\,dx = -\int_{\mathbb{R}^n} u(x)(D_j\varphi(x))\,dx \tag{2.15}$$

if $u \in \mathcal{C}^1_0(\mathbb{R}^n)$. Thus if $u \in \mathcal{S}'(\mathbb{R}^n)$ again we *define*

$$D_j u(\psi) = -u(D_j\psi) \;\forall\; \psi \in \mathcal{S}(\mathbb{R}^n).$$

Then it is clear that $D_j u \in \mathcal{S}'(\mathbb{R}^n)$.

Iterating these definition we find that D^α, for any multi-index α, defines a linear map

$$D^\alpha : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,. \tag{2.16}$$

In general a linear differential operator with constant coefficients is a sum of such "monomials". For example Laplace's operator is

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2} = D_1^2 + D_2^2 + \cdots + D_n^2\,.$$

We will be interested in trying to solve differential equations such as

$$\Delta u = f \in \mathcal{S}'(\mathbb{R}^n)\,.$$

We can also multiply $u \in \mathcal{S}'(\mathbb{R}^n)$ by $\varphi \in \mathcal{S}(\mathbb{R}^n)$, simply defining

$$\varphi u(\psi) = u(\varphi\psi) \;\forall\; \psi \in \mathcal{S}(\mathbb{R}^n). \tag{2.17}$$

For this to make sense it suffices to check that

$$\sum_{\substack{|\alpha|\le k,\\|\beta|\le k}} \sup_{\mathbb{R}^n} \left|x^\alpha D^\beta(\varphi\psi)\right| \le C \sum_{\substack{|\alpha|\le k,\\|\beta|\le k}} \sup_{\mathbb{R}^n} \left|x^\alpha D^\beta\psi\right| . \tag{2.18}$$

This follows easily from Leibniz' formula.

Now, to start thinking of $u \in \mathcal{S}'(\mathbb{R}^n)$ as a generalized function we first define its *support*. Recall that

$$\operatorname{supp}(\psi) = \operatorname{clos}\{x \in \mathbb{R}^n; \psi(x) \neq 0\} . \tag{2.19}$$

We can write this in another 'weak' way which is easier to generalize. Namely

$$p \notin \operatorname{supp}(u) \Leftrightarrow \ \exists \varphi \in \mathcal{S}(\mathbb{R}^n) \,,\ \varphi(p) \neq 0 \,,\ \varphi u = 0 \,. \tag{2.20}$$

In fact this definition makes sense for *any* $u \in \mathcal{S}'(\mathbb{R}^n)$.

LEMMA 2.5. *The set* $\operatorname{supp}(u)$ *defined by* (2.20) *is a closed subset of* $\mathbb{R}^n$ *and reduces to* (2.19) *if* $u \in \mathcal{S}(\mathbb{R}^n)$.

PROOF. The set defined by (2.20) is closed, since

$$\operatorname{supp}(u)^{\complement} = \{p \in \mathbb{R}^n;\ \exists\ \varphi \in \mathcal{S}(\mathbb{R}^n),\ \varphi(p) \neq 0,\ \varphi u = 0\} \tag{2.21}$$

is clearly open — the same φ works for nearby points. If $\psi \in \mathcal{S}(\mathbb{R}^n)$ we define $u_\psi \in \mathcal{S}'(\mathbb{R}^n)$, which we will again identify with ψ, by

$$u_\psi(\varphi) = \int \varphi(x)\psi(x)\,dx \,. \tag{2.22}$$

Obviously $u_\psi = 0 \Longrightarrow \psi = 0$, simply set $\varphi = \overline{\psi}$ in (2.22). Thus the map

$$\mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto u_\psi \in \mathcal{S}'(\mathbb{R}^n) \tag{2.23}$$

is injective. We want to show that

$$\operatorname{supp}(u_\psi) = \operatorname{supp}(\psi) \tag{2.24}$$

on the left given by (2.20) and on the right by (2.19). We show first that

$$\operatorname{supp}(u_\psi) \subset \operatorname{supp}(\psi).$$

Thus, we need to see that $p \notin \operatorname{supp}(\psi) \Rightarrow p \notin \operatorname{supp}(u_\psi)$. The first condition is that $\psi(x) = 0$ in a neighbourhood, U of p, hence there is a $\mathcal{C}^\infty$ function φ with support in U and $\varphi(p) \neq 0$. Then $\varphi\psi \equiv 0$. Conversely suppose $p \notin \operatorname{supp}(u_\psi)$. Then there exists $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi(p) \neq 0$ and $\varphi u_\psi = 0$, i.e., $\varphi u_\psi(\eta) = 0\ \forall\ \eta \in \mathcal{S}(\mathbb{R}^n)$. By the injectivity of $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ this means $\varphi\psi = 0$, so $\psi \equiv 0$ in a neighborhood of p and $p \notin \operatorname{supp}(\psi)$. $\square$

Consider the simplest examples of distribution which are not functions, namely those with support at a given point p. The obvious one is the Dirac delta 'function'

$$\delta_p(\varphi) = \varphi(p) \ \forall \ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{2.25}$$

We can make many more, because D^α is *local*

$$\operatorname{supp}(D^\alpha u) \subset \operatorname{supp}(u) \ \forall \ u \in \mathcal{S}'(\mathbb{R}^n)\,. \tag{2.26}$$

Indeed, $p \notin \operatorname{supp}(u) \Rightarrow \exists \ \varphi \in \mathcal{S}(\mathbb{R}^n)$, $\varphi u \equiv 0$, $\varphi(p) \neq 0$. Thus each of the distributions $D^\alpha \delta_p$ also has support contained in $\{p\}$. In fact none of them vanish, and they are all linearly independent.

3. Convolution and density

We have defined an inclusion map

$$\mathcal{S}(\mathbb{R}^n) \ni \varphi \longmapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n),\ u_\varphi(\psi) = \int_{\mathbb{R}^n} \varphi(x)\psi(x)\,dx \ \forall \ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{3.1}$$

This allows us to 'think of' $\mathcal{S}(\mathbb{R}^n)$ as a subspace of $\mathcal{S}'(\mathbb{R}^n)$; that is we habitually identify u_φ with φ. We can do this because we know (3.1) to be injective. We can extend the map (3.1) to include bigger spaces

$$\begin{gathered}
\mathcal{C}^0_0(\mathbb{R}^n) \ni \varphi \longmapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n) \\
L^p(\mathbb{R}^n) \ni \varphi \longmapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n) \\
M(\mathbb{R}^n) \ni \mu \longmapsto u_\mu \in \mathcal{S}'(\mathbb{R}^n) \\
u_\mu(\psi) = \int_{\mathbb{R}^n} \psi\,d\mu\,,
\end{gathered} \tag{3.2}$$

but we need to know that these maps are injective before we can forget about them.

We can see this using *convolution*. This is a sort of 'product' of functions. To begin with, suppose $v \in \mathcal{C}^0_0(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. We define a new function by 'averaging v with respect to ψ:'

$$v * \psi(x) = \int_{\mathbb{R}^n} v(x-y)\psi(y)\,dy\,. \tag{3.3}$$

The integral converges by dominated convergence, namely $\psi(y)$ is integrable and v is bounded,

$$|v(x-y)\psi(y)| \le \|v\|_{\mathcal{C}^0_0}\ |\psi(y)|\ .$$

We can use the same sort of estimates to show that $v*\psi$ is continuous. Fix $x\in\mathbb{R}^n$,

$$v*\psi(x+x')-v*\psi(x) = \int (v(x+x'-y)-v(x-y))\psi(y)\,dy\,. \tag{3.4}$$

To see that this is small for x' small, we split the integral into two pieces. Since ψ is very small near infinity, given $\epsilon>0$ we can choose R so large that

$$\|v\|_\infty\cdot\int_{|y]|\ge R}|\psi(y)|\;dy\le\epsilon/4\,. \tag{3.5}$$

The set $|y|\le R$ is compact and if $|x|\le R'$, $|x'|\le 1$ then $|x+x'-y|\le R+R'+1$. A continuous function is *uniformly continuous* on any compact set, so we can chose $\delta>0$ such that

$$\sup_{\substack{|x'|<\delta\\ |y|\le R}}|v(x+x'-y)-v(x-y)|\cdot\int_{|y|\le R}|\psi(y)|\;dy\;<\epsilon/2\,. \tag{3.6}$$

Combining (3.5) and (3.6) we conclude that $v*\psi$ is continuous. Finally, we conclude that

$$v\in\mathcal{C}^0_0(\mathbb{R}^n)\Rightarrow v*\psi\in\mathcal{C}^0_0(\mathbb{R}^n)\,. \tag{3.7}$$

For this we need to show that $v*\psi$ is small at infinity, which follows from the fact that v is small at infinity. Namely given $\epsilon>0$ there exists $R>0$ such that $|v(y)|\le\epsilon$ if $|y|\ge R$. Divide the integral defining the convolution into two

$$\begin{aligned}|v*\psi(x)|\le\int_{|y|>R}u(y)\psi(x-y)dy+\int_{y<R}|u(y)\psi(x-y)|dy\\ \le\epsilon/2\|\psi\|_\infty+\|u\|_\infty\sup_{B(x,R)}|\psi|.\end{aligned}$$

Since $\psi\in\mathcal{S}(\mathbb{R}^n)$ the last constant tends to 0 as $|x|\to\infty$.

We can do much better than this! Assuming $|x'|\le 1$ we can use Taylor's formula with remainder to write

$$\psi(z+x')-\psi(z)=\int_0^{\prime}\frac{d}{dt}\psi(z+tx')\,dt=\sum_{j=1}^n x_j\cdot\tilde\psi_j(z,x')\,. \tag{3.8}$$

As Problem 23 I ask you to check carefully that

$$\psi_j(z;x')\in\mathcal{S}(\mathbb{R}^n)\text{ depends continuously on }x'\text{ in }\;|x'|\le 1\,. \tag{3.9}$$

Going back to (3.3))we can use the translation and reflection-invariance of Lebesgue measure to rewrite the integral (by changing variable) as

$$v * \psi(x) = \int_{\mathbb{R}^n} v(y)\psi(x-y)\,dy\,. \tag{3.10}$$

This reverses the role of v and ψ and shows that if *both* v and ψ are in $\mathcal{S}(\mathbb{R}^n)$ then $v * \psi = \psi * v$.

Using this formula on (3.4) we find

$$\begin{aligned} v * \psi(x+x') - v * \psi(x) &= \int v(y)(\psi(x+x'-y) - \psi(x-y))\,dy \\ &= \sum_{j=1}^n x_j \int_{\mathbb{R}^n} v(y)\tilde{\psi}_j(x-y,x')\,dy = \sum_{j=1}^n x_j (v * \psi_j(\cdot;x')(x)\,. \end{aligned} \tag{3.11}$$

From (3.9) and what we have already shown, $v * \psi(\cdot;x')$ is continuous in both variables, and is in $\mathcal{C}_0^0(\mathbb{R}^n)$ in the first. Thus

$$v \in \mathcal{C}_0^0(\mathbb{R}^n)\,,\ \psi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow v * \psi \in \mathcal{C}_0^1(\mathbb{R}^n)\,. \tag{3.12}$$

In fact we also see that

$$\frac{\partial}{\partial x_j} v * \psi = v * \frac{\partial \psi}{\partial x_j}\,. \tag{3.13}$$

Thus $v * \psi$ inherits its regularity from ψ.

Proposition 3.1. *If $v \in \mathcal{C}_0^0(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$v * \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n) = \bigcap_{k \geq 0} \mathcal{C}_0^k(\mathbb{R}^n)\,. \tag{3.14}$$

Proof. This follows from (3.12), (3.13) and induction. □

Now, let us make a more special choice of ψ. We have shown the existence of

$$\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)\,,\ \varphi \geq 0\,,\ \operatorname{supp}(\varphi) \subset \{|x| \leq 1\}\,. \tag{3.15}$$

We can also assume $\int_{\mathbb{R}^n} \varphi\,dx = 1$, by multiplying by a positive constant. Now consider

$$\varphi_t(x) = t^{-n}\varphi\left(\frac{x}{t}\right)\ 1 \geq t > 0\,. \tag{3.16}$$

This has all the same properties, except that

$$\operatorname{supp}\varphi_t \subset \{|x| \leq t\}\,,\ \int \varphi_t\,dx = 1\,. \tag{3.17}$$

Proposition 3.2. *If $v \in \mathcal{C}_0^0(\mathbb{R}^n)$ then as $t \to 0$, $v_t = v * \varphi_t \to v$ in $\mathcal{C}_0^0(\mathbb{R}^n)$.*

PROOF. using (3.17) we can write the difference as

$$|v_t(x) - v(x)| = |\int_{\mathbb{R}^n} (v(x-y) - v(x))\varphi_t(y)\,dy| \\ \le \sup_{|y|\le t} |v(x-y) - v(x)| \to 0. \tag{3.18}$$

Here we have used the fact that $\varphi_t \ge 0$ has support in $|y| \le t$ and has integral 1. Thus $v_t \to v$ uniformly on any set on which v is uniformly continuous, namel $\mathbb{R}^n$! □

COROLLARY 3.3. *$\mathcal{C}_0^k(\mathbb{R}^n)$ is dense in $\mathcal{C}_0^p(\mathbb{R}^n)$ for any $k \ge p$.*

PROPOSITION 3.4. *$\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{C}_0^k(\mathbb{R}^n)$ for any $k \ge 0$.*

PROOF. Take $k = 0$ first. The subspace $\mathcal{C}_c^0(\mathbb{R}^n)$ is dense in $\mathcal{C}_0^0(\mathbb{R}^n)$, by cutting off outside a large ball. If $v \in \mathcal{C}_c^0(\mathbb{R}^n)$ has support in $\{|x| \le R\}$ then

$$v * \varphi_t \in \mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$$

has support in $\{|x| \le R+1\}$. Since $v * \varphi_t \to v$ the result follows for $k = 0$.

For $k \ge 1$ the same argument works, since $D^\alpha(v * \varphi_t) = (D^\alpha V) * \varphi_t$. □

COROLLARY 3.5. *The map from finite Radon measures*

$$M_{fin}(\mathbb{R}^n) \ni \mu \longmapsto u_\mu \in \mathcal{S}'(\mathbb{R}^n) \tag{3.19}$$

is injective.

Now, we want the same result for $L^2(\mathbb{R}^n)$ (and maybe for $L^p(\mathbb{R}^n)$, $1 \le p < \infty$). I leave the measure-theoretic part of the argument to you.

PROPOSITION 3.6. *Elements of $L^2(\mathbb{R}^n)$ are "continuous in the mean" i.e.,*

$$\lim_{|t|\to 0} \int_{\mathbb{R}^n} |u(x+t) - u(x)|^2 \, dx = 0\,. \tag{3.20}$$

This is Problem 24.

Using this we conclude that

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \text{ is dense} \tag{3.21}$$

as before. First observe that the space of L^2 functions of compact support is dense in $L^2(\mathbb{R}^n)$, since

$$\lim_{R\to\infty} \int_{|x|\ge R} |u(x)|^2 \, dx = 0 \;\forall\; u \in L^2(\mathbb{R}^n)\,.$$

Then look back at the discussion of $v * \varphi$, now v is replaced by $u \in L^2_c(\mathbb{R}^n)$. The compactness of the support means that $u \in L^1(\mathbb{R}^n)$ so in

$$u * \varphi(x) = \int_{\mathbb{R}^n} u(x-y)\varphi(y)dy \tag{3.22}$$

the integral is absolutely convergent. Moreover

$$\begin{aligned}
&|u * \varphi(x+x') - u * \varphi(x)| \\
&\qquad = \left| \int u(y)(\varphi(x+x'-y) - \varphi(x-y))\, dy \right| \\
&\qquad\qquad \leq C\|u\| \sup_{|y|\leq R} |\varphi(x+x'-y) - \varphi(x-y)| \to 0
\end{aligned}$$

when $\{|x| \leq R\}$ large enough. Thus $u * \varphi$ is continuous and the same argument as before shows that

$$u * \varphi_t \in \mathcal{S}(\mathbb{R}^n)\,.$$

Now to see that $u * \varphi_t \to u$, assuming u has compact support (or not) we estimate the integral

$$\begin{aligned}
|u * \varphi_t(x) - u(x)| &= \left| \int (u(x-y) - u(x))\varphi_t(y)\, dy \right| \\
&\leq \int |u(x-y) - u(x)|\, \varphi_t(y)\, dy\,.
\end{aligned}$$

Using the same argument twice

$$\begin{aligned}
&\int |u * \varphi_t(x) - u(x)|^2\, dx \\
&\quad \leq \iiint |u(x-y) - u(x)|\, \varphi_t(y)\, |u(x-y') - u(x)|\, \varphi_t(y')\, dx\, dy\, dy' \\
&\qquad \leq \left(\int |u(x-y) - u(x)|^2\, \varphi_t(y)\varphi_t(y') dx\, dy\, dy' \right) \\
&\qquad\qquad \leq \sup_{|y|\leq t} \int |u(x-y) - u(x)|^2\, dx\,.
\end{aligned}$$

Note that at the second step here I have used Schwarz's inequality with the integrand written as the product

$$|u(x-y) - u(x)|\, \varphi_t^{1/2}(y)\varphi_t^{1/2}(y') \cdot |u(x-y') - u(x)|\, \varphi_t^{1/2}(y)\varphi_t^{1/2}(y')\,.$$

Thus we now know that

$$L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ is injective.}$$

This means that all our usual spaces of functions 'sit inside' $\mathcal{S}'(\mathbb{R}^n)$.

Finally we can use convolution with φ_t to show the existence of *smooth* partitions of unity. If $K \Subset U \subset \mathbb{R}^n$ is a compact set in an open set then we have shown the existence of $\xi \in \mathcal{C}^0_c(\mathbb{R}^n)$, with $\xi = 1$ in some neighborhood of K and $\xi = 1$ in some neighborhood of K and $\operatorname{supp}(\xi) \Subset U$.

Then consider $\xi * \varphi_t$ for t small. In fact

$$\operatorname{supp}(\xi * \varphi_t) \subset \{p \in \mathbb{R}^n\,;\, \operatorname{dist}(p, \operatorname{supp}\xi) \le 2t\}$$

and similarly, $0 \le \xi * \varphi_t \le 1$ and

$$\xi * \varphi_t = 1 \text{ at } p \text{ if } \xi = 1 \text{ on } B(p, 2t)\,.$$

Using this we get:

Proposition 3.7. *If $U_a \subset \mathbb{R}^n$ are open for $a \in A$ and $K \Subset \bigcup_{a\in A} U_a$ then there exist finitely many $\varphi_i \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, with $0 \le \varphi_i \le 1$, $\operatorname{supp}(\varphi_i) \subset U_{a_i}$ such that $\sum\limits_i \varphi_i = 1$ in a neighbourhood of K.*

Proof. By the compactness of K we may choose a finite open subcover. Using Lemma 15.7 we may choose a continuous partition, ϕ'_i, of unity subordinate to this cover. Using the convolution argument above we can replace ϕ'_i by $\phi'_i * \varphi_t$ for $t > 0$. If t is sufficiently small then this is again a partition of unity subordinate to the cover, but now smooth. □

Next we can make a simple 'cut off argument' to show

Lemma 3.8. *The space $\mathcal{C}^\infty_c(\mathbb{R}^n)$ of $\mathcal{C}^\infty$ functions of compact support is dense in $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Choose $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\varphi(x) = 1$ in $|x| \le 1$. Then given $\psi \in \mathcal{S}(\mathbb{R}^n)$ consider the sequence

$$\psi_n(x) = \varphi(x/n)\psi(x)\,.$$

Clearly $\psi_n = \psi$ on $|x| \le n$, so if it converges in $\mathcal{S}(\mathbb{R}^n)$ it must converge to ψ. Suppose $m \ge n$ then by Leibniz's formula[4]

$$D^\alpha_x(\psi_n(x) - \psi_m(x)) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta_x \left(\varphi(\frac{x}{n}) - \varphi(\frac{x}{m})\right) \cdot D^{\alpha-\beta}_x \psi(x)\,.$$

All derivatives of $\varphi(x/n)$ are bounded, independent of n and $\psi_n = \psi_m$ in $|x| \le n$ so for any p

$$|D^\alpha_x(\psi_n(x) - \psi_m(x))| \le \begin{cases} 0 & |x| \le n \\ C_{\alpha,p}\langle x\rangle^{-2p} & |x| \ge n \end{cases}\,.$$

[4]Problem 25.

Hence ψ_n is Cauchy in $\mathcal{S}(\mathbb{R}^n)$. $\square$

Thus every element of $\mathcal{S}'(\mathbb{R}^n)$ is determined by its restriction to $\mathcal{C}_c^\infty(\mathbb{R}^n)$. The support of a tempered distribution was defined above to be

$$\text{supp}(u) = \{x \in \mathbb{R}^n;\ \exists\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,,\ \varphi(x) \neq 0\,,\ \varphi u = 0\}^{\complement}\,. \tag{3.23}$$

Using the preceding lemma and the construction of smooth partitions of unity we find

Proposition 3.9. *f* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\text{supp}(u) = \emptyset$ *then* $u = 0$.

Proof. From (3.23), if $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\text{supp}(\psi u) \subset \text{supp}(u)$. If $x \ni \text{supp}(u)$ then, by definition, $\varphi u = 0$ for some $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi(x) \neq 0$. Thus $\varphi \neq 0$ on $B(x, \epsilon)$ for $\epsilon > 0$ sufficiently small. If $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ has support in $B(x, \epsilon)$ then $\psi u = \tilde{\psi}\varphi u = 0$, where $\tilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{R}^n)$:

$$\tilde{\psi} = \begin{cases} \psi/\varphi & \text{in } B(x, \epsilon) \\ 0 & \text{elsewhere}\,. \end{cases}$$

Thus, given $K \Subset \mathbb{R}^n$ we can find $\varphi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, supported in such balls, so that $\sum_j \varphi_j \equiv 1$ on K but $\varphi_j u = 0$. For given $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ apply this to $\text{supp}(\mu)$. Then

$$\mu = \sum_j \varphi_j \mu \Rightarrow u(\mu) = \sum_j (\phi_j u)(\mu) = 0\,.$$

Thus $u = 0$ on $\mathcal{C}_c^\infty(\mathbb{R}^n)$, so $u = 0$. $\square$

The linear space of distributions of compact support will be denoted $\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$; it is often written $\mathcal{E}'(\mathbb{R}^n)$.

Now let us give a characterization of the 'delta function'

$$\delta(\varphi) = \varphi(0)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,,$$

or at least the one-dimensional subspace of $\mathcal{S}'(\mathbb{R}^n)$ it spans. This is based on the simple observation that $(x_j\varphi)(0) = 0$ if $\varphi \in \mathcal{S}(\mathbb{R}^n)$!

Proposition 3.10. *If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *satisfies* $x_j u = 0$, $j = 1, \cdots, n$ *then* $u = c\delta$.

Proof. The main work is in characterizing the null space of δ as a linear functional, namely in showing that

$$\mathcal{H} = \{\varphi \in \mathcal{S}(\mathbb{R}^n);\ \varphi(0) = 0\} \tag{3.24}$$

can also be written as

$$\mathcal{H} = \left\{\varphi \in \mathcal{S}(\mathbb{R}^n);\ \varphi = \sum_{j=1}^n x_j \psi_j\,,\ \varphi_j \in \mathcal{S}(\mathbb{R}^n)\right\}\,. \tag{3.25}$$

Clearly the right side of (3.25) is contained in the left. To see the converse, suppose first that

$$\varphi \in \mathcal{S}(\mathbb{R}^n)\,,\ \varphi = 0 \text{ in } |x| < 1\,. \tag{3.26}$$

Then define

$$\psi = \begin{cases} 0 & |x| < 1 \\ \varphi/\left|x\right|^2 & |x| \ge 1\,. \end{cases}$$

All the derivatives of $1/\left|x\right|^2$ are bounded in $|x| \ge 1$, so from Leibniz's formula it follows that $\psi \in \mathcal{S}(\mathbb{R}^n)$. Since

$$\varphi = \sum_j x_j(x_j\psi)$$

this shows that φ of the form (3.26) is in the right side of (3.25). In general suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned} \varphi(x) - \varphi(0) &= \int_0^t \frac{d}{dt}\varphi(tx)\,dt \\ &= \sum_{j=1}^n x_j \int_0^t \frac{\partial\varphi}{\partial x_j}(tx)\,dt\,. \end{aligned} \tag{3.27}$$

Certainly these integrals are $\mathcal{C}^\infty$, but they may not decay rapidly at infinity. However, choose $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\mu = 1$ in $|x| \le 1$. Then (3.27) becomes, if $\varphi(0) = 0$,

$$\begin{aligned} \varphi &= \mu\varphi + (1-\mu)\varphi \\ &= \sum_{j=1}^n x_j\psi_j + (1-\mu)\varphi\,,\ \psi_j = \mu \int_0^t \frac{\partial\varphi}{\partial x_j}(tx)\,dt \in \mathcal{S}(\mathbb{R}^n)\,. \end{aligned}$$

Since $(1-\mu)\varphi$ is of the form (3.26), this proves (3.25).

Our assumption on u is that $x_j u = 0$, thus

$$u(\varphi) = 0 \ \forall \ \varphi \in \mathcal{H}$$

by (3.25). Choosing μ as above, a general $\varphi \in \mathcal{S}(\mathbb{R}^n)$ can be written

$$\varphi = \varphi(0)\cdot\mu + \varphi'\,,\ \varphi' \in \mathcal{H}\,.$$

Then

$$u(\varphi) = \varphi(0)u(\mu) \Rightarrow u = c\delta\,,\ c = u(\mu)\,.$$

$\square$

This result is quite powerful, as we shall soon see. The Fourier transform of an element $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is[5]

$$\hat{\varphi}(\xi) = \int e^{-ix\cdot\xi}\varphi(x)\,dx\,,\ \xi \in \mathbb{R}^n\,. \tag{3.28}$$

The integral certainly converges, since $|\varphi| \le C\langle x\rangle^{-n-1}$. In fact it follows easily that $\hat{\varphi}$ is continuous, since

$$\begin{aligned}|\hat{\varphi}(\xi) - \hat{\varphi}(\xi')| \in \int \left|e^{ix-\xi} - e^{-x\cdot\xi'}\right| |\varphi|\ dx \\ \to 0 \text{ as } \xi' \to \xi\,.\end{aligned}$$

In fact

Proposition 3.11. *Fourier transformation,* (3.28)*, defines a continuous linear map*

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,,\ \mathcal{F}\varphi = \hat{\varphi}\,. \tag{3.29}$$

Proof. Differentiating under the integral[6] sign shows that

$$\partial_{\xi_j}\hat{\varphi}(\xi) = -i\int e^{-ix\cdot\xi}x_j\varphi(x)\,dx\,.$$

Since the integral on the right *is* absolutely convergent that shows that (remember the i's)

$$D_{\xi_j}\hat{\varphi} = -\widehat{x_j\varphi}\,,\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{3.30}$$

Similarly, if we multiply by ξ_j and observe that $\xi_j e^{-ix\cdot\xi} = i\frac{\partial}{\partial x_j}e^{-ix\cdot\xi}$ then integration by parts shows

$$\begin{aligned}\xi_j\hat{\varphi} &= i\int (\frac{\partial}{\partial x_j}e^{-ix\cdot\xi})\varphi(x)\,dx \\ &= -i\int e^{-ix\cdot\xi}\frac{\partial\varphi}{\partial x_j}\,dx \\ \widehat{D_j\varphi} &= \xi_j\hat{\varphi}\,,\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,.\end{aligned} \tag{3.31}$$

Since $x_j\varphi$, $D_j\varphi \in \mathcal{S}(\mathbb{R}^n)$ these results can be iterated, showing that

$$\xi^\alpha D_\xi^\beta\hat{\varphi} = \mathcal{F}\left((-1)^{|\beta|}D^\alpha{}_x x^\beta\varphi\right)\,. \tag{3.32}$$

Thus $\left|\xi^\alpha D_\xi^\beta\hat{\varphi}\right| \le C_{\alpha\beta}\sup\left|\langle x\rangle^{+n+1}D^\alpha{}_x x^\beta\varphi\right| \le C\|\langle x\rangle^{n+1+|\beta|}\varphi\|_{\mathcal{C}^{|\alpha|}}$, which shows that $\mathcal{F}$ is continuous as a map (3.32).

□

[5]Normalizations vary, but it doesn't matter much.

[6]See [**6**]

Suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Since $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ we can consider the distribution $u \in \mathcal{S}'(\mathbb{R}^n)$

$$u(\varphi) = \int_{\mathbb{R}^n} \hat{\varphi}(\xi)\, d\xi\,. \tag{3.33}$$

The continuity of u follows from the fact that integration is continuous and (3.29). Now observe that

$$\begin{aligned} u(x_j\varphi) &= \int_{\mathbb{R}^n} \widehat{x_j\varphi}(\xi)\, d\xi \\ &= -\int_{\mathbb{R}^n} D_{\xi_j}\hat{\varphi}\, d\xi = 0 \end{aligned}$$

where we use (3.30). Applying Proposition 3.10 we conclude that $u = c\delta$ for some (universal) constant c. By definition this means

$$\int_{\mathbb{R}^n} \hat{\varphi}(\xi)\, d\xi = c\varphi(0)\,. \tag{3.34}$$

So what is the constant? To find it we need to work out an example. The simplest one is

$$\varphi = \exp(-|x|^2/2)\,.$$

LEMMA 3.12. *The Fourier transform of the Gaussian* $\exp(-|x|^2/2)$ *is the Gaussian* $(2\pi)^{n/2}\exp(-|\xi|^2/2)$.

PROOF. There are two obvious methods — one uses complex analysis (Cauchy's theorem) the other, which I shall follow, uses the uniqueness of solutions to ordinary differential equations.

First observe that $\exp(-|x|^2/2) = \prod_j \exp(-x_j^2/2)$. Thus[7]

$$\hat{\varphi}(\xi) = \prod_{j=1}^{n} \hat{\psi}(\xi_j)\,,\ \psi(x) = e^{-x^2/2}\,,$$

being a function of one variable. Now ψ satisfies the differential equation

$$(\partial_x + x)\,\psi = 0\,,$$

and is the *only* solution of this equation up to a constant multiple. By (3.30) and (3.31) its Fourier transform satisfies

$$\widehat{\partial_x\psi} + \widehat{x\psi} = i\xi\hat{\psi} + i\frac{d}{d\xi}\hat{\varphi} = 0\,.$$

[7] Really by Fubini's theorem, but here one can use Riemann integrals.

This is the same equation, but in the ξ variable. Thus $\hat{\psi} = ce^{-|\xi|^2/2}$. Again we need to find the constant. However,

$$\hat{\psi}(0) = c = \int e^{-x^2/2}\,dx = (2\pi)^{1/2}$$

by the standard use of polar coordinates:

$$c^2 = \int_{\mathbb{R}^n} e^{-(x^2+y^2)/2}\,dx\,dy = \int_0^\infty \int_0^{2\pi} e^{-r^2/2} r\,dr\,d\theta = 2\pi\,.$$

This proves the lemma.

□

Thus we have shown that for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \hat{\varphi}(\xi)\,d\xi = (2\pi)^n \varphi(0)\,. \tag{3.35}$$

Since this is true for $\varphi = \exp(-|x|^2/2)$. The identity allows us to *invert* the Fourier transform.

4. Fourier inversion

It is shown above that the Fourier transform satisfies the identity

$$\varphi(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\varphi}(\xi)\,d\xi \;\forall\; \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{4.1}$$

If $y \in \mathbb{R}^n$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ set $\psi(x) = \varphi(x+y)$. The translation-invariance of Lebesgue measure shows that

$$\begin{aligned}\hat{\psi}(\xi) &= \int e^{-ix\cdot\xi} \varphi(x+y)\,dx\\ &= e^{iy\cdot\xi}\hat{\varphi}(\xi)\,.\end{aligned}$$

Applied to ψ the inversion formula (4.1) becomes

$$\begin{aligned}\varphi(y) = \psi(0) &= (2\pi)^{-n} \int \hat{\psi}(\xi)\,d\xi\\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\cdot\xi}\hat{\varphi}(\xi)\,d\xi\,.\end{aligned} \tag{4.2}$$

THEOREM 4.1. *Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isomorphism with inverse*

$$\mathcal{G} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,,\; \mathcal{G}\psi(y) = (2\pi)^{-n} \int e^{iy\cdot\xi}\psi(\xi)\,d\xi\,. \tag{4.3}$$

PROOF. The identity (4.2) shows that $\mathcal{F}$ is $1-1$, i.e., injective, since we can remove φ from $\hat{\varphi}$. Moreover,

$$\mathcal{G}\psi(y) = (2\pi)^{-n}\mathcal{F}\psi(-y) \tag{4.4}$$

So $\mathcal{G}$ is also a continuous linear map, $\mathcal{G} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Indeed the argument above shows that $\mathcal{G} \circ \mathcal{F} = Id$ and the same argument, with some changes of sign, shows that $\mathcal{F} \cdot \mathcal{G} = Id$. Thus F and $\mathcal{G}$ are isomorphisms.

□

LEMMA 4.2. *For all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, Paseval's identity holds:*

$$\int_{\mathbb{R}^n} \varphi\overline{\psi}\, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\varphi}\overline{\hat{\psi}}\, d\xi\,. \tag{4.5}$$

PROOF. Using the inversion formula on φ,

$$\begin{aligned}\int \varphi\overline{\psi}\, dx &= (2\pi)^{-n} \int \left(e^{ix\cdot\xi}\hat{\varphi}(\xi)\, d\xi\right) \overline{\psi}(x)\, dx \\ &= (2\pi)^{-n} \int \hat{\varphi}(\xi) \overline{\int e^{-ix\cdot\xi}\psi(x)\, dx}\, d\xi \\ &= (2\pi)^{-n} \int \hat{\varphi}(\xi)\overline{\hat{\varphi}}(\xi)\, d\xi\,.\end{aligned}$$

Here the integrals are absolutely convergent, justifying the exchange of orders.

□

PROPOSITION 4.3. *Fourier transform extends to an isomorphism*

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,. \tag{4.6}$$

PROOF. Setting $\varphi = \psi$ in (4.5) shows that

$$\|\mathcal{F}\varphi\|_{L^2} = (2\pi)^{n/2}\|\varphi\|_{L^2}\,. \tag{4.7}$$

In particular this proves, given the known density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, that $\mathcal{F}$ is an isomorphism, with inverse $\mathcal{G}$, as in (4.6).

□

For any $m \in \mathbb{R}$

$$\langle x\rangle^m L^2(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n)\,;\, \langle x\rangle^{-m}\hat{u} \in L^2(\mathbb{R}^n)\right\}$$

is a well-defined subspace. We define the *Sobolev spaces* on $\mathbb{R}^n$ by, for $m \geq 0$

$$H^m(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n)\,;\, \hat{u} = \mathcal{F}u \in \langle\xi\rangle^{-m}L^2(\mathbb{R}^n)\right\}\,. \tag{4.8}$$

$$\text{Thus } H^m(\mathbb{R}^n) \subset H^{m'}(\mathbb{R}^n) \text{ if } m \geq m'\,,\ H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)\,.$$

LEMMA 4.4. *If $m \in \mathbb{N}$ is an integer, then*

$$u \in H^m(\mathbb{R}^n) \Leftrightarrow D^\alpha u \in L^2(\mathbb{R}^n) \ \forall \ |\alpha| \le m\,. \tag{4.9}$$

PROOF. By definition, $u \in H^m(\mathbb{R}^n)$ implies that $\langle\xi\rangle^{-m}\hat{u} \in L^2(\mathbb{R}^n)$. Since $\widehat{D^\alpha u} = \xi^\alpha \hat{u}$ this certainly implies that $D^\alpha u \in L^2(\mathbb{R}^n)$ for $|\alpha| \le m$. Conversely if $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le m$ then $\xi^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le m$ and since

$$\langle\xi\rangle^m \le C_m \sum_{|\alpha|\le m} |\xi^\alpha|\ .$$

this in turn implies that $\langle\xi\rangle^m \hat{u} \in L^2(\mathbb{R}^n)$.

□

Now that we have considered the Fourier transform of Schwartz test functions we can use the usual method, of duality, to extend it to tempered distributions. If we set $\eta = \overline{\hat{\psi}}$ then $\hat{\psi} = \overline{\eta}$ and $\psi = \mathcal{G}\hat{\psi} = \mathcal{G}\overline{\eta}$ so

$$\overline{\psi}(x) = (2\pi)^{-n}\int e^{-ix\cdot\xi}\overline{\hat{\psi}}(\xi)\,d\xi \\ = (2\pi)^{-n}\int e^{-ix\cdot\xi}\eta(\xi)\,d\xi = (2\pi)^{-n}\hat{\eta}(x).$$

Substituting in (4.5) we find that

$$\int \varphi\hat{\eta}\,dx = \int \hat{\varphi}\eta\,d\xi\,.$$

Now, recalling how we embed $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ we see that

$$u_{\hat{\varphi}}(\eta) = u_\varphi(\hat{\eta}) \ \forall\ \eta \in \mathcal{S}(\mathbb{R}^n)\,. \tag{4.10}$$

DEFINITION 4.5. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ we* define *its Fourier transform by*

$$\hat{u}(\varphi) = u(\hat{\varphi}) \ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{4.11}$$

As a composite map, $\hat{u} = u \cdot \mathcal{F}$, with each term continuous, $\hat{u}$ is continuous, i.e., $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$.

PROPOSITION 4.6. *The definition* (4.7) *gives an isomorphism*

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,,\ \mathcal{F}u = \hat{u}$$

satisfying the identities

$$\widehat{D^\alpha u} = \xi^\alpha u\,,\ \widehat{x^\alpha u} = (-1)^{|\alpha|}D^\alpha \hat{u}\,. \tag{4.12}$$

Proof. Since $\hat{u} = u \circ \mathcal{F}$ and $\mathcal{G}$ is the 2-sided inverse of $\mathcal{F}$,

$$u = \hat{u} \circ \mathcal{G} \tag{4.13}$$

gives the inverse to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, showing it to be an isomorphism. The identities (4.12) follow from their counterparts on $\mathcal{S}(\mathbb{R}^n)$:

$$\begin{aligned}\widehat{D^\alpha u}(\varphi) &= D^\alpha u(\hat{\varphi}) = u((-1)^{|\alpha|} D^\alpha \hat{\varphi}) \\ &= u(\widehat{\xi^\alpha \varphi}) = \hat{u}(\xi^\alpha \varphi) = \xi^\alpha \hat{u}(\varphi) \ \forall \ \varphi \in \mathcal{S}(\mathbb{R}^n)\,.\end{aligned}$$

□

We can also define Sobolev spaces of *negative* order:

$$H^m(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n)\,;\, \hat{u} \in \langle \xi \rangle^{-m} L^2(\mathbb{R}^n) \right\}\,. \tag{4.14}$$

Proposition 4.7. *If $m \le 0$ is an integer then $u \in H^m(\mathbb{R}^n)$ if and only if it can be written in the form*

$$u = \sum_{|\alpha| \le -m} D^\alpha v_\alpha\,,\ v_\alpha \in L^2(\mathbb{R}^n)\,. \tag{4.15}$$

Proof. If $u \in \mathcal{S}'(\mathbb{R}^n)$ is of the form (4.15) then

$$\hat{u} = \sum_{|\alpha| \le -m} \xi^\alpha \hat{v}_\alpha \text{ with } \hat{v}\alpha \in L^2(\mathbb{R}^n)\,. \tag{4.16}$$

Thus $\langle \xi \rangle^m \hat{u} = \sum_{|\alpha| \le -m} \xi^\alpha \langle \xi \rangle^m \hat{v}_\alpha$. Since all the factors $\xi^\alpha \langle \xi \rangle^m$ are bounded, each term here is in $L^2(\mathbb{R}^n)$, so $\langle \xi \rangle^m \hat{u} \in L^2(\mathbb{R}^n)$ which is the definition, $u \in \langle \xi \rangle^{-m} L^2(\mathbb{R}^n)$.

Conversely, suppose $u \in H^m(\mathbb{R}^n)$, i.e., $\langle \xi \rangle^m \hat{u} \in L^2(\mathbb{R}^n)$. The function

$$\left(\sum_{|\alpha| \le -m} |\xi^\alpha| \right) \cdot \langle \xi \rangle^m \in L^2(\mathbb{R}^n) \ (m < 0)$$

is bounded below by a positive constant. Thus

$$v = \left(\sum_{|\alpha| \le -m} |\xi^\alpha| \right)^{-1} \hat{u} \in L^2(\mathbb{R}^n)\,.$$

Each of the functions $\hat{v}_\alpha = \operatorname{sgn}(\xi^\alpha) \hat{v} \in L^2(\mathbb{R}^n)$ so the identity (4.16), and hence (4.15), follows with these choices.

□

PROPOSITION 4.8. *Each of the Sobolev spaces $H^m(\mathbb{R}^n)$ is a Hilbert space with the norm and inner product*

$$\|u\|_{H^m} = \left(\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \langle\xi\rangle^{2m}\, d\xi\right)^{1/2}, \tag{4.17}$$

$$\langle u, v\rangle = \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}\langle\xi\rangle^{2m}\, d\xi\,.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^m(\mathbb{R}^n)$ is dense for each m and the pairing

$$H^m(\mathbb{R}^n) \times H^{-m}(\mathbb{R}^n) \ni (u, u') \longmapsto \tag{4.18}$$

$$((u, u')) = \int_{\mathbb{R}^n} \hat{u}'(\xi)\hat{u}'(\cdot\xi)\, d\xi \in \mathbb{C}$$

gives an identification $(H^m(\mathbb{R}^n))' = H^{-m}(\mathbb{R}^n)$.

PROOF. The Hilbert space property follows essentially directly from the definition (4.14) since $\langle\xi\rangle^{-m}L^2(\mathbb{R}^n)$ is a Hilbert space with the norm (4.17). Similarly the density of $\mathcal{S}$ in $H^m(\mathbb{R}^n)$ follows, since $\mathcal{S}(\mathbb{R}^n)$ dense in $L^2(\mathbb{R}^n)$ (Problem L11.P3) implies $\langle\xi\rangle^{-m}\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$ is dense in $\langle\xi\rangle^{-m}L^2(\mathbb{R}^n)$ and so, since $\mathcal{F}$ is an isomorphism in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R}^n)$.

Finally observe that the pairing in (4.18) makes sense, since $\langle\xi\rangle^{-m}\hat{u}(\xi)$, $\langle\xi\rangle^m\hat{u'}(\xi) \in L^2(\mathbb{R}^n)$ implies

$$\hat{u}(\xi))\hat{u'}(-\xi) \in L^1(\mathbb{R}^n)\,.$$

Furthermore, by the self-duality of $L^2(\mathbb{R}^n)$ each continuous linear functional

$$U : H^m(\mathbb{R}^n) \to \mathbb{C}\,,\ U(u) \le C\|u\|_{H^m}$$

can be written uniquely in the form

$$U(u) = ((u, u')) \text{ for some } u' \in H^{-m}(\mathbb{R}^n)\,.$$

□

Notice that if u, $u' \in \mathcal{S}(\mathbb{R}^n)$ then

$$((u, u')) = \int_{\mathbb{R}^n} u(x)u'(x)\, dx\,.$$

This is always how we "pair" functions — it is the natural pairing on $L^2(\mathbb{R}^n)$. Thus in (4.18) what we have shown is that this pairing on test function

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (u, u') \longmapsto ((u, u')) = \int_{\mathbb{R}^n} u(x)u'(x)\, dx$$

extends by *continuity* to $H^m(\mathbb{R}^n) \times H^{-m}(\mathbb{R}^n)$ (for each fixed m) when it identifies $H^{-m}(\mathbb{R}^n)$ as the dual of $H^m(\mathbb{R}^n)$. This was our 'picture' at the beginning.

For $m > 0$ the spaces $H^m(\mathbb{R}^n)$ represents elements of $L^2(\mathbb{R}^n)$ that have "m" derivatives in $L^2(\mathbb{R}^n)$. For $m < 0$ the elements are ?? of "up to $-m$" derivatives of L^2 functions. For integers this is precisely ??.

5. Sobolev embedding

The properties of Sobolev spaces are briefly discussed above. If m is a positive integer then $u \in H^m(\mathbb{R}^n)$ 'means' that u has up to m derivatives in $L^2(\mathbb{R}^n)$. The question naturally arises as to the sense in which these 'weak' derivatives correspond to old-fashioned 'strong' derivatives. Of course when m is not an integer it is a little harder to imagine what these 'fractional derivatives' are. However the main result is:

THEOREM 5.1 (Sobolev embedding). *If $u \in H^m(\mathbb{R}^n)$ where $m > n/2$ then $u \in \mathcal{C}_0^0(\mathbb{R}^n)$, i.e.,*

$$H^m(\mathbb{R}^n) \subset \mathcal{C}_0^0(\mathbb{R}^n)\,,\ m > n/2\,. \tag{5.1}$$

PROOF. By definition, $u \in H^m(\mathbb{R}^n)$ means $v \in \mathcal{S}'(\mathbb{R}^n)$ and $\langle\xi\rangle^m \hat{u}(\xi) \in L^2(\mathbb{R}^n)$. Suppose first that $u \in \mathcal{S}(\mathbb{R}^n)$. The Fourier inversion formula shows that

$$\begin{aligned}(2\pi)^n\,|u(x)| &= \left|\int e^{ix\cdot\xi}\hat{u}(\xi)\,d\xi\right| \\ &\le \left(\int_{\mathbb{R}^n}\langle\xi\rangle^{2m}\,|\hat{u}(\xi)|^2\,d\xi\right)^{1/2}\cdot\left(\sum_{\mathbb{R}^n}\langle\xi\rangle^{-2m}\,d\xi\right)^{1/2}.\end{aligned}$$

Now, if $m > n/2$ then the second integral is finite. Since the first integral is the norm on $H^m(\mathbb{R}^n)$ we see that

$$\sup_{\mathbb{R}^n}|u(x)| = \|u\|_{L^\infty} \le (2\pi)^{-n}\|u\|_{H^m}\,,\ m > n/2\,. \tag{5.2}$$

This is all for $u \in \mathcal{S}(\mathbb{R}^n)$, but $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^m(\mathbb{R}^n)$ is dense. The estimate (5.2) shows that if $u_j \to u$ in $H^m(\mathbb{R}^n)$, with $u_j \in \mathcal{S}(\mathbb{R}^n)$, then $u_j \to u'$ in $\mathcal{C}_0^0(\mathbb{R}^n)$. In fact $u' = u$ in $\mathcal{S}'(\mathbb{R}^n)$ since $u_j \to u$ in $L^2(\mathbb{R}^n)$ and $u_j \to u'$ in $\mathcal{C}_0^0(\mathbb{R}^n)$ both imply that $\int u_j\varphi$ converges, so

$$\int_{\mathbb{R}^n} u_j\varphi \to \int_{\mathbb{R}^n} u\varphi = \int_{\mathbb{R}^n} u'\varphi\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

$\square$

Notice here the precise meaning of $u = u'$, $u \in H^m(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, $u' \in \mathcal{C}_0^0(\mathbb{R}^n)$. When identifying $u \in L^2(\mathbb{R}^n)$ with the corresponding tempered distribution, the values on any set of measure zero 'are lost'. Thus as *functions* (5.1) means that each $u \in H^m(\mathbb{R}^n)$ has a representative $u' \in \mathcal{C}_0^0(\mathbb{R}^n)$.

We can extend this to higher derivatives by noting that

PROPOSITION 5.2. *If $u \in H^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, then $D^\alpha u \in H^{m-|\alpha|}(\mathbb{R}^n)$ and*

$$D^\alpha : H^m(\mathbb{R}^n) \to H^{m-|\alpha|}(\mathbb{R}^n) \tag{5.3}$$

is continuous.

PROOF. First it is enough to show that each D_j defines a continuous linear map

$$D_j : H^m(\mathbb{R}^n) \to H^{m-1}(\mathbb{R}^n) \ \forall\ j \tag{5.4}$$

since then (5.3) follows by composition.

If $m \in \mathbb{R}$ then $u \in H^m(\mathbb{R}^n)$ means $\hat{u} \in \langle\xi\rangle^{-m} L^2(\mathbb{R}^n)$. Since $\widehat{D_j u} = \xi_j \cdot \hat{u}$, and

$$|\xi_j|\,\langle\xi\rangle^{-m} \le C_m \langle\xi\rangle^{-m+1} \ \forall\ m$$

we conclude that $D_j u \in H^{m-1}(\mathbb{R}^n)$ and

$$\|D_j u\|_{H^{m-1}} \le C_m \|u\|_{H^m}\,.$$

$\square$

Applying this result we see

COROLLARY 5.3. *If $k \in \mathbb{N}_0$ and $m > \frac{n}{2} + k$ then*

$$H^m(\mathbb{R}^n) \subset \mathcal{C}_0^k(\mathbb{R}^n)\,. \tag{5.5}$$

PROOF. If $|\alpha| \le k$, then $D^\alpha u \in H^{m-k}(\mathbb{R}^n) \subset \mathcal{C}_0^0(\mathbb{R}^n)$. Thus the 'weak derivatives' $D^\alpha u$ are continuous. Still we have to check that this means that u is itself k times continuously differentiable. In fact this again follows from the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^m(\mathbb{R}^n)$. The continuity in (5.3) implies that if $u_j \to u$ in $H^m(\mathbb{R}^n)$, $m > \frac{n}{2} + k$, then $u_j \to u'$ in $\mathcal{C}_0^k(\mathbb{R}^n)$ (using its completeness). However $u = u'$ as before, so $u \in \mathcal{C}_0^k(\mathbb{R}^n)$.

$\square$

In particular we see that

$$H^\infty(\mathbb{R}^n) = \bigcap_m H^m(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)\,. \tag{5.6}$$

These functions are not in general Schwartz test functions.

PROPOSITION 5.4. *Schwartz space can be written in terms of weighted Sobolev spaces*

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_k \langle x\rangle^{-k} H^k(\mathbb{R}^n)\,. \tag{5.7}$$

PROOF. This follows directly from (5.5) since the left side is contained in

$$\bigcap_k \langle x\rangle^{-k} \mathcal{C}_0^{k-n}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n).$$

□

THEOREM 5.5 (Schwartz representation). *Any tempered distribution can be written in the form of a finite sum*

$$u = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} x^\alpha D_x^\beta u_{\alpha\beta}\,,\ u_{\alpha\beta} \in \mathcal{C}_0^0(\mathbb{R}^n). \tag{5.8}$$

or *in the form*

$$u = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} D_x^\beta (x^\alpha v_{\alpha\beta}),\ \ v_{\alpha\beta} \in \mathcal{C}_0^0(\mathbb{R}^n). \tag{5.9}$$

Thus every tempered distribution is a finite sum of derivatives of continuous functions of poynomial growth.

PROOF. Essentially by definition any $u \in \mathcal{S}'(\mathbb{R}^n)$ is continuous with respect to *one* of the norms $\|\langle x\rangle^k \varphi\|_{\mathcal{C}^k}$. From the Sobolev embedding theorem we deduce that, with $m > k + n/2$,

$$|u(\varphi)| \le C \|\langle x\rangle^k \varphi\|_{H^m}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

This is the same as

$$\left|\langle x\rangle^{-k} u(\varphi)\right| \le C\|\varphi\|_{H^m}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

which shows that $\langle x\rangle^{-k} u \in H^{-m}(\mathbb{R}^n)$, i.e., from Proposition 4.8,

$$\langle x\rangle^{-k} u = \sum_{|\alpha|\le m} D^\alpha u_\alpha\,,\ \ u_\alpha \in L^2(\mathbb{R}^n)\,.$$

In fact, choose $j > n/2$ and consider $v_\alpha \in H^j(\mathbb{R}^n)$ defined by $\hat{v}_\alpha = \langle \xi\rangle^{-j} \hat{u}_\alpha$. As in the proof of Proposition 4.14 we conclude that

$$u_\alpha = \sum_{|\beta|\le j} D^\beta u'_{\alpha,\beta}\,,\ u'_{\alpha,\beta} \in H^j(\mathbb{R}^n) \subset \mathcal{C}_0^0(\mathbb{R}^n)\,.$$

Thus,[8]

$$u = \langle x\rangle^k \sum_{|\gamma|\le M} D^\gamma_\alpha v_\gamma\,,\ \ v_\gamma \in \mathcal{C}^0_0(\mathbb{R}^n)\,. \tag{5.10}$$

To get (5.9) we 'commute' the factor $\langle x\rangle^k$ to the inside; since I have not done such an argument carefully so far, let me do it as a lemma.

LEMMA 5.6. *For any $\gamma \in \mathbb{N}^n_0$ there are polynomials $p_{\alpha,\gamma}(x)$ of degrees at most $|\gamma-\alpha|$ such that*

$$\langle x\rangle^k D^\gamma v = \sum_{\alpha\le\gamma} D^{\gamma-\alpha}\left(p_{\alpha,\gamma}\langle x\rangle^{k-2|\gamma-\alpha|}v\right)\,.$$

PROOF. In fact it is convenient to prove a more general result. Suppose p is a polynomial of a degree at most j then there exist polynomials of degrees at most $j+|\gamma-\alpha|$ such that

$$p\langle x\rangle^k D^\gamma v = \sum_{\alpha\le\gamma} D^{\gamma-\alpha}(p_{\alpha,\gamma}\langle x\rangle^{k-2|\gamma-\alpha|}v)\,. \tag{5.11}$$

The lemma follows from this by taking $p=1$.

Furthermore, the identity (5.11) is trivial when $\gamma=0$, and proceeding by induction we can suppose it is known whenever $|\gamma|\le L$. Taking $|\gamma| = L+1$,

$$D^\gamma = D_j D^{\gamma'}\ \ |\gamma'| = L.$$

Writing the identity for γ' as

$$p\langle x\rangle^k D^{\gamma'} = \sum_{\alpha'\le\gamma'} D^{\gamma'-\alpha'}(p_{\alpha',\gamma'}\langle x\rangle^{k-2|\gamma'-\alpha'|}v)$$

we may differentiate with respect to x_j. This gives

$$\begin{aligned} p\langle x\rangle^k D^\gamma &= -D_j(p\langle x\rangle^k)\cdot D^{\gamma'}v \\ &+ \sum_{|\alpha'|\le\gamma} D^{\gamma-\alpha'}(p'_{\alpha',\gamma'}\langle x\rangle^{k-2|\gamma-\alpha|+2}v)\,. \end{aligned}$$

The first term on the right expands to

$$(-(D_jp)\cdot\langle x\rangle^k D^{\gamma'}v - \frac{1}{i}kpx_j\langle x\rangle^{k-2}D^{\gamma'}v)\,.$$

We may apply the inductive hypothesis to each of these terms and rewrite the result in the form (5.11); it is only necessary to check the order of the polynomials, and recall that $\langle x\rangle^2$ is a polynomial of degree 2. $\square$

[8]This is probably the most useful form of the representation theorem!

Applying Lemma 5.6 to (5.10) gives (5.9), once negative powers of $\langle x\rangle$ are absorbed into the continuous functions. Then (5.8) follows from (5.9) and Leibniz's formula. □

The question arises as to the 'meaning' of the fractional derivatives implicit in the definition of $H^s(\mathbb{R}^n)$ when $s > 0$ but is not an integer. We know from that if k is the integral part of s then $u \in H^s(\mathbb{R}^n)$ is equivalent to the statement that $D^\alpha u \in H^{s-k}(\mathbb{R}^n)$, $|\alpha| \le k$. So we can concentrate on the case $s \in (0, 1)$.

Proposition 5.7. *If* $0 < s < 1$ *then* $u \in H^s(\mathbb{R}^n)$ *if and only if* $u \in L^2(\mathbb{R}^n)$ *and*

$$\iint \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy < \infty. \tag{5.12}$$

Proof. If $u \in L^2(\mathbb{R}^n)$ the integrand in (5.12) is a non-negative measureable function so the finiteness of the integral is a well-defined condition. In fact the part of the integral away from the diagonal, $x = y$, is already finite – if $c > 0$ then

$$\iint_{|x-y|>c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy < \infty. \tag{5.13}$$

To see this use the inequality, $|u(x) - u(y)|^2 \le 2|u(x)|^2 + 2|u(y|^2$ giving two integrals so that after changing variables and using Fubini's theorem

$$\iint_{|x-y|>c} \frac{|u(x)|^2}{|x-y|^{n+2s}} dxdy = \int |u(x)|^2 dx \int_{|z|>c} |z|^{-n-2s} dz \tag{5.14}$$

where both factors are finite. Thus the significance of (5.12) is in the convergence across the diagonal.

Now, if $u \in \mathcal{S}(\mathbb{R}^n)$ then (5.12) does indeed hold. We have just seen the convergence when $|x-y| > c$ and in $|x-y| < c$ Taylor's formula (or the mean value theorem) gives, in view of the rapid decay of the derivative

$$|u(x) - u(y)| \le C|x-y|(1+|x|)^{-n},\ |x-y| \le c \tag{5.15}$$

so this part of the integral is also finite

$$\iint_{|x-y|<c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \le C \int (1+|x|)^{-2n-21} \int_{|z|<c} |z|^{-n-2s+2} \tag{5.16}$$

and since the power of $|z|$ is strictly larger than $-n$ the integral converges across $|z| = 0$.

So, now consider the integral (5.12) when $u \in \mathcal{S}(\mathbb{R}^n)$; we have just seen that it is a well-defined Lebesgue integral. We can change variable to give, again by Fubini (which tells us that the first integral converges a.e. and the result is integrable)

$$\int dz|z|^{-n-2s}\int |u(z+y)-u(y)|^2dy.$$

Then we use Plancherels' formula on the inner integral to write it as

(5.17)
$$\int |u(z+y)-u(y)|^2dy = (2\pi)^{-n}\int |\mathcal{F}(u(z+\cdot)-u(\cdot))|^2d\xi,$$
$$\mathcal{F}(u(z+\cdot)-u(\cot)) = (e^{z\cdot\xi}-1)\hat{u}(\xi) \Longrightarrow$$
$$\int dz|z|^{-n-2s}\int |u(z+y)-u(y)|^2dy = \int d\xi F(\xi)|\hat{u}(\xi)|^2,\ F(\xi) = \int \frac{|e^{iz\cdot\xi}-1|^2}{|z|^{n+2s}}dz.$$

As it must by Fubini's theorem, the integrand defining $F(\xi)$ does indeed converge. Near infinity the integrand is bounded by $2|z|^{-n-2s}$ which is integrable and near zero, by Taylor's formula, it is bounded by $C|z|^{-n-2s+2}$ which is also integrable. Furthermore it is clearly rotation-invariant. Applying an orthogonal transformation $F(O\xi) = F(\xi)$ using the change of variable to O^tz. Thus in fact $F(\xi) = F(|\xi|)$. It is also homogeneous of degree $2s$ as can be seen by scaling the variable. Thus in fact

$$F(\xi) = c|\xi|^{2s},\ c>0. \tag{5.18}$$

So in fact for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\iint \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy = c\int |\xi|^{2s}|\hat{u}(\xi)|^2d\xi. \tag{5.19}$$

Since $1+|\xi|^2s$ is bounded above and below by positive multiples of $(1+|\xi|^2)^s$

$$\left(\|u\|^2_{L^2} + \iint \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\right)^{\frac{1}{2}}$$

is a Hilbert norm which is equivalent to the H^s norm on $\mathcal{S}(\mathbb{R}^n)$.

So this proves the result; the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$ means that if $u \in H^s(\mathbb{R}^n)$ then we can find a sequence $u_n \in \mathcal{S}(\mathbb{R}^n)$ such that $u_n \to u$ in $L^2(\mathbb{R}^n)$ and u_n converges in $H^s(\mathbb{R}^n)$ (to u of course). This implies the convergence of the integral (5.12) for u_n as $n \to \infty$ and hence that the integrals for u over $|x-y| > \delta$ are bounded by a fixed constant. This, but monotone convergence implies that the integral for u is finite and conversely and that (5.19) holds in the limit. $\square$

6. Differential operators.

In the last third of the course we will apply what we have learned about distributions, and a little more, to understand properties of differential operators with constant coefficients. Before I start talking about these, I want to prove another density result.

So far we have *not* defined a topology on $\mathcal{S}'(\mathbb{R}^n)$ – I will leave this as an optional exercise.[9] However we shall consider a notion of convergence. Suppose $u_j \in \mathcal{S}'(\mathbb{R}^n)$ is a sequence in $\mathcal{S}'(\mathbb{R}^n)$. It is said to *converge weakly* to $u \in \mathcal{S}'(\mathbb{R}^n)$ if

$$u_j(\varphi) \to u(\varphi) \ \forall \ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{6.1}$$

There is no 'uniformity' assumed here, it is rather like pointwise convergence (except the linearity of the functions makes it seem stronger).

PROPOSITION 6.1. *The subspace $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is weakly dense, i.e., each $u \in \mathcal{S}'(\mathbb{R}^n)$ is the weak limit of a subspace $u_j \in \mathcal{S}(\mathbb{R}^n)$.*

PROOF. We can use Schwartz representation theorem to write, for some m depending on u,

$$u = \langle x\rangle^m \sum_{|\alpha|\le m} D^\alpha u_\alpha\,,\ u_\alpha \in L^2(\mathbb{R}^n)\,.$$

We know that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, in the sense of metric spaces so we can find $u_{\alpha,j} \in \mathcal{S}(\mathbb{R}^n)$, $u_{\alpha,j} \to u_\alpha$ in $L^2(\mathbb{R}^n)$. The density result then follows from the basic properties of weak convergence. □

PROPOSITION 6.2. *If $u_j \to u$ and $u'_j \to u'$ weakly in $\mathcal{S}'(\mathbb{R}^n)$ then $cu_j \to cu$, $u_j + u'_j \to u + u'$, $D^\alpha u_j \to D^\alpha u$ and $\langle x\rangle^m u_j \to \langle x\rangle^m u$ weakly in $\mathcal{S}'(\mathbb{R}^n)$.*

PROOF. This follows by writing everyting in terms of pairings, for example if $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$D^\alpha u_j(\varphi) = u_j((-1)^{(\alpha)} D^\alpha \varphi) \to u((-1)^{(\alpha)} D^\alpha \varphi) = D^\alpha u(\varphi)\,.$$

□

This weak density shows that our definition of D_j, and $x_j\times$ are unique if we require Proposition 6.2 to hold.

We have discussed differentiation as an operator (meaning just a linear map between spaces of function-like objects)

$$D_j : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,.$$

[9] Problem 34.

Any polynomial on $\mathbb{R}^n$

$$p(\xi) = \sum_{|\alpha| \le m} p_\alpha \xi^\alpha \, , \; p_\alpha \in \mathbb{C}$$

defines a differential operator[10]

$$p(D)u = \sum_{|\alpha| \le m} p_\alpha D^\alpha u \, . \tag{6.2}$$

Before discussing any general theorems let me consider some examples.

$$\text{On } \mathbb{R}^2, \; \overline{\partial} = \partial_x + i\partial_y \text{“d-bar operator”} \tag{6.3}$$

$$\text{on } \mathbb{R}^n, \; \Delta = \sum_{j=1}^n D_j^2 \text{“Laplacian”} \tag{6.4}$$

$$\text{on } \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}, \; D_t^2 - \Delta \text{“Wave operator”} \tag{6.5}$$

$$\text{on} \mathbb{R} \times \mathbb{R}^{\mathrm{n}} = \mathbb{R}^{\mathrm{n}+1}, \; \partial_{\mathrm{t}} + \Delta \text{“Heat operator”} \tag{6.6}$$

$$\text{on } \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}, \; D_t + \Delta \text{“Schrödinger operator”} \tag{6.7}$$

Functions, or distributions, satisfying $\overline{\partial} u = 0$ are said to be *holomorphic*, those satisfying $\Delta u = 0$ are said to be *harmonic*.

DEFINITION 6.3. *An element $E \in \mathcal{S}'(\mathbb{R}^n)$ satisfying*

$$P(D)E = \delta \tag{6.8}$$

is said *to be a (tempered) fundamental solution of $P(D)$.*

THEOREM 6.4 (without proof). *Every non-zero constant coefficient differential operator has a tempered fundamental solution.*

This is quite hard to prove and not as interetsing as it might seem. We will however give lots of examples, starting with $\overline{\partial}$. Consider the function

$$E(x, y) = \frac{1}{2\pi}(x + iy)^{-1} \, , \; (x, y) \neq 0 \, . \tag{6.9}$$

LEMMA 6.5. *$E(x, y)$ is locally integrable and so defines $E \in \mathcal{S}'(\mathbb{R}^2)$ by*

$$E(\varphi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} (x + iy)^{-1} \varphi(x, y) \, dx \, dy \, , \tag{6.10}$$

and E so defined is a tempered fundamental solution of $\overline{\partial}$.

[10] More correctly a partial differential operator with constant coefficients.

PROOF. Since $(x+iy)^{-1}$ is smooth and bounded away from the origin the local integrability follows from the estimate, using polar coordinates,

$$\int_{|(x,y)|\le 1} \frac{dx\,dy}{|x+iy|} = \int_0^{2\pi}\int_0^1 \frac{r\,dr\,d\theta}{r} = 2\pi\,. \tag{6.11}$$

Differentiating directly in the region where it is smooth,

$$\partial_x(x+iy)^{-1} = -(x+iy)^{-2}\,,\ \partial_y(x+iy)^{-1} = -i(x\in iy)^{-2}$$

so indeed, $\overline{\partial}E = 0$ in $(x,y)\neq 0$.[11]

The derivative is *really* defined by

$$\begin{aligned}(\overline{\partial}E)(\varphi) &= E(-\overline{\partial}\varphi)\\ &= \lim_{\epsilon\downarrow 0} -\frac{1}{2\pi}\int_{\substack{|x|\ge\epsilon\\|y|\ge\epsilon}}(x+iy)^{-1}\ \overline{\partial}\varphi\,dx\,dy\,.\end{aligned} \tag{6.12}$$

Here I have cut the space $\{|x|\le\epsilon\,,\ |y|\le\epsilon\}$ out of the integral and used the local integrability in taking the limit as $\epsilon\downarrow 0$. Integrating by parts in x we find

$$\begin{aligned}-\int_{\substack{|x|\ge\epsilon\\|y|\ge\epsilon}}(x+iy)^{-1}\partial_x\varphi\,dx\,dy &= \int_{\substack{|x|\ge\epsilon\\|y|\ge\epsilon}}(\partial_x(x+iy)^{-1})\varphi\,dx\,dy\\ &+\int_{\substack{|y|\le\epsilon\\x=\epsilon}}(x+iy)^{-1}\varphi(x,y)\,dy - \int_{\substack{|y|\le\epsilon\\x=-\epsilon}}(x+iy)^{-1}\varphi(x,y)\,dy\,.\end{aligned}$$

There is a corrsponding formula for integration by parts in y so, recalling that $\overline{\partial}E = 0$ away from $(0,0)$,

$$\begin{aligned}2\pi\overline{\partial}E(\varphi) &= \\ &\lim_{\epsilon\downarrow 0}\int_{|y|\le\epsilon}[(\epsilon+iy)^{-1}\varphi(\epsilon,y) - (-\epsilon+iy)^{-1}\varphi(-\epsilon,y)]\,dy\\ &\qquad + i\lim_{\epsilon\downarrow 0}\int_{|x|\le\epsilon}[(x+i\epsilon)^{-1}\varphi(x,\epsilon) - (x-i\epsilon)^{-1}\varphi(x,\epsilon)]\,dx\,,\end{aligned} \tag{6.13}$$

assuming that both limits exist. Now, we can write

$$\varphi(x,y) = \varphi(0,0) + x\psi_1(x_1 y) + y\psi_2(x,y)\,.$$

Replacing φ by either $x\psi_1$ or $y\psi_2$ in (6.13) both limits are zero. For example

$$\left|\int_{|y|\le\epsilon}(\epsilon+iy)^{-1}\epsilon\psi_1(\epsilon,y)\,dy\right| \le \int_{|y|\le\epsilon}|\psi_1|\to 0\,.$$

[11]Thus at this stage we know $\overline{\partial}E$ must be a sum of derivatives of δ.

Thus we get the same result in (6.13) by replacing $\varphi(x,y)$ by $\varphi(0,0)$. Then $2\pi\overline{\partial}E(\varphi) = c\varphi(0)$,

$$c = \lim_{\epsilon\downarrow 0} 2\epsilon \int_{|y|\le\epsilon} \frac{dy}{\epsilon^2+y^2} = \lim_{\epsilon\downarrow 0} < \int_{|y|\le 1} \frac{dy}{1+y^2} = 2\pi\,.$$

□

Let me remind you that we have already discussed the convolution of functions

$$u * v(x) = \int u(x-y)v(y)\,dy = v * u(x)\,.$$

This makes sense provided u is of slow growth and $s \in \mathcal{S}(\mathbb{R}^n)$. In fact we can rewrite the definition in terms of pairing

$$(u*\varphi)(x) = \langle u, \varphi(x-\cdot)\rangle \tag{6.14}$$

where the $\cdot$ indicates the variable in the pairing.

Theorem 6.6 (Hörmander, Theorem 4.1.1). *If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $u*\varphi \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$ and if* $\operatorname{supp}(\varphi) \Subset \mathbb{R}^n$

$$\operatorname{supp}(u*\varphi) \subset \operatorname{supp}(u) + \operatorname{supp}(\varphi)\,.$$

For any multi-index α

$$D^\alpha(u*\varphi) = D^\alpha u * \varphi = u * D^\alpha\varphi\,.$$

Proof. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then for any fixed $x \in \mathbb{R}^n$,

$$\varphi(x-\cdot) \in \mathcal{S}(\mathbb{R}^n)\,.$$

Indeed the seminorm estimates required are

$$\sup_y (1+|y|^2)^{k/2}\,|D^\alpha{}_y\varphi(x-y)| < \infty\ \forall\ \alpha, k > 0\,.$$

Since $D^\alpha{}_y\varphi(x-y) = (-1)^{|\alpha|}(D^\alpha\varphi)(x-y)$ and

$$(1+|y|^2) \le (1+|x-y|^2)(1+|x|^2)$$

we conclude that

$$\|(1+|y|^2)^{k/2}D^\alpha{}_y(x-y)\|_{L^\infty} \le (1+|x|^2)^{k/2}\|\langle y\rangle^k D^\alpha{}_y\varphi(y)\|_{L^\infty}\,.$$

The continuity of $u \in \mathcal{S}'(\mathbb{R}^n)$ means that for some k

$$|u(\varphi)| \le C \sup_{|\alpha|\le k} \|(y)^k D^\alpha\varphi\|_{L^\infty}$$

so it follows that

$$|u*\varphi(x)| = |\langle u, \varphi(x-\cdot)\rangle| \le C(1+|x|^2)^{k/2}\,. \tag{6.15}$$

The argument above shows that $x \mapsto \varphi(x - \cdot)$ is a continuous function of $x \in \mathbb{R}^n$ with values in $\mathcal{S}(\mathbb{R}^n)$, so $u * \varphi$ is continuous and satisfies (6.15). It is therefore an element of $\mathcal{S}'(\mathbb{R}^n)$.

Differentiability follows in the same way since for each j, with e_j the jth unit vector

$$\frac{\varphi(x + se_j - y) - \varphi(x - y)}{s} \in \mathcal{S}(\mathbb{R}^n)$$

is continuous in $x \in \mathbb{R}^n$, $s \in \mathbb{R}$. Thus, $u * \varphi$ has continuous partial derivatives and

$$D_j u * \varphi = u * D_j \varphi \,.$$

The same argument then shows that $u * \varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$. That $D_j(u * \varphi) = D_j u * \varphi$ follows from the definition of derivative of distributions

$$\begin{aligned} D_j(u * \varphi(x)) &= (u * D_j \varphi)(x) \\ &= \langle u, D_{x_j} \varphi(x - y) \rangle = -\langle u(y), D_{y_j} \varphi(x - y) \rangle_y \\ &= (D_j u) * \varphi \,. \end{aligned}$$

Finally consider the support property. Here we are assuming that $\operatorname{supp}(\varphi)$ is compact; we also know that $\operatorname{supp}(u)$ is a closed set. We have to show that

$$\overline{x} \notin \operatorname{supp}(u) + \operatorname{supp}(\varphi) \tag{6.16}$$

implies $u * \varphi(x') = 0$ for x' near $\overline{x}$. Now (6.16) just means that

$$\operatorname{supp} \varphi(\overline{x} - \cdot) \cap \operatorname{supp}(u) = \phi \,, \tag{6.17}$$

Since $\operatorname{supp} \varphi(x - \cdot) = \{y \in \mathbb{R}^n; x - y \in \operatorname{supp}(\varphi)\}$, so both statements mean that there is *no* $y \in \operatorname{supp}(\varphi)$ with $\overline{x} - y \in \operatorname{supp}(u)$. This can also be written

$$\operatorname{supp}(\varphi) \cap \operatorname{supp} u(x - \cdot) = \phi$$

and as we showed when discussing supports implies

$$u * \varphi(x') = \langle u(x' - \cdot), \varphi \rangle = 0 \,.$$

From (6.17) this is an *open* condition on x', so the support property follows.

□

Now suppose φ, $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$(u * \varphi) * \psi = u * (\varphi * \psi) \,. \tag{6.18}$$

This is really Hörmander's Lemma 4.1.3 and Theorem 4.1.2; I ask you to prove it as Problem 35.

We have shown that $u * \varphi$ is $\mathcal{C}^\infty$ if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, i.e., the regularity of $u * \varphi$ follows from the regularity of *one* of the factors. This makes it reasonable to expect that $u * v$ can be defined when $u \in \mathcal{S}'(\mathbb{R}^n)$, $v \in \mathcal{S}'(\mathbb{R}^n)$ and one of them has compact support. If $v \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then

$$u * v(\varphi) = \int \langle u(\cdot), v(x - \cdot)\rangle \varphi(x)\, dx = \int \langle u(\cdot), v(x - \cdot)\rangle \check{v}\varphi(-x)\, dx$$

where $\check{\varphi}(z) = \varphi(-z)$. In fact using Problem 35,

$$u * v(\varphi) = ((u * v) * \check{\varphi})(0) = (u * (v * \check{\varphi}))(0)\,. \tag{6.19}$$

Here, v, φ are both smooth, but notice

Lemma 6.7. *If $v \in \mathcal{S}'(\mathbb{R}^n)$ has compact support and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $v * \varphi \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. Since $v \in \mathcal{S}'(\mathbb{R}^n)$ has compact support there exists $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\chi v = v$. Then

$$\begin{aligned} v * \varphi(x) &= (\chi v) * \varphi(x) = \langle \chi v(y), \varphi(x - y)\rangle_y \\ &= \langle u(y), \chi(y)\varphi(x - y)\rangle_y\,. \end{aligned}$$

Thus, for some k,

$$|v * \varphi(x)| \le C\|\chi(y)\varphi(x - y)\|_{(k)}$$

where $\|\ \|_{(k)}$ is one of our norms on $\mathcal{S}(\mathbb{R}^n)$. Since χ is supported in some large ball,

$$\begin{aligned} &\|\chi(y)\varphi(x - y)\|_{(k)} \\ &\qquad \le \sup_{|\alpha| \le k} \left|\langle y\rangle^k D^\alpha{}_y(\chi(y)\varphi(x - y))\right| \\ &\qquad \le C \sup_{|y| \le R} \sup_{|\alpha| \le k} |(D^\alpha\varphi)(x - y)| \\ &\qquad \le C_N \sup_{|y| \le R} (1 + |x - y|^2)^{-N/2} \\ &\qquad \le C_N (1 + |x|^2)^{-N/2}\,. \end{aligned}$$

Thus $(1 + |x|^2)^{N/2}\,|v * \varphi|$ is bounded for each N. The same argument applies to the derivative using Theorem 6.6, so

$$v * \varphi \in \mathcal{S}(\mathbb{R}^n)\,.$$

$\square$

In fact we get a little more, since we see that for each k there exists k' and C (depending on k and v) such that

$$\|v * \varphi\|_{(k)} \le C\|\varphi\|_{(k')}\,.$$

This means that

$$v* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is a continuous linear map.

Now (6.19) allows us to define $u*v$ when $u \in \mathcal{S}'(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ has compact support by

$$u * v(\varphi) = u * (v * \check{\varphi})(0)\,.$$

Using the continuity above, I ask you to check that $u * v \in \mathcal{S}'(\mathbb{R}^n)$ in Problem 36. For the moment let me assume that this convolution has the same properties as before – I ask you to check the main parts of this in Problem 37.

Recall that $E \in \mathcal{S}'(\mathbb{R}^n)$ is a fundamental situation for $P(D)$, a constant coefficient differential operator, if $P(D)E = \delta$. We also use a weaker notion.

Definition 6.8. *A* parametrix *for a constant coefficient differential operator $P(D)$ is a distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$P(D)F = \delta + \psi\,,\ \ \psi \in \mathcal{C}^\infty(\mathbb{R}^n)\,. \tag{6.20}$$

An operator $P(D)$ is said to be hypoelliptic *if it has a parametrix satisfying*

$$\operatorname{sing\,supp}(F) \subset \{0\}\ , \tag{6.21}$$

where for any $u \in \mathcal{S}'(\mathbb{R}^n)$

$$(\operatorname{sing\,supp}(u))^{\complement} = \{\overline{x} \in \mathbb{R}^n;\ \exists\, \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)\,, \\ \varphi(\overline{x}) \neq 0, \varphi u \in \mathcal{C}^\infty_c(\mathbb{R}^n)\}\ . \tag{6.22}$$

Since the same φ must work for nearby points in (6.22), the set $\operatorname{sing\,supp}(u)$ is *closed*. Furthermore

$$\operatorname{sing\,supp}(u) \subset \operatorname{supp}(u)\,. \tag{6.23}$$

As Problem 37 I ask you to show that if $K \Subset \mathbb{R}^n$ and $K \cap \operatorname{sing\,supp}(u) = \phi$ the $\exists\, \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\varphi(x) = 1$ in a neighbourhood of K such that $\varphi u \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. In particular

$$\operatorname{sing\,supp}(u) = \phi \Rightarrow u \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)\,. \tag{6.24}$$

Theorem 6.9. *If $P(D)$ is hypoelliptic then*

$$\operatorname{sing\,supp}(u) = \operatorname{sing\,supp}(P(D)u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n)\,. \tag{6.25}$$

Proof. One half of this is true for *any* differential operator:

Lemma 6.10. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ then for any polynomial*

$$\operatorname{sing\,supp}(P(D)u) \subset \operatorname{sing\,supp}(u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n)\,. \tag{6.26}$$

□

PROOF. We must show that $\overline{x} \notin \operatorname{sing\,supp}(u) \Rightarrow \overline{x} \notin \operatorname{sing\,supp}(P(D)u)$. Now, if $\overline{x} \notin \operatorname{sing\,supp}(u)$ we can find $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\varphi \equiv 1$ near $\overline{x}$, such that $\varphi u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned} P(D)u &= P(D)(\varphi u + (1-\varphi)u) \\ &= P(D)(\varphi u) + P(D)((1-\varphi)u)\,. \end{aligned}$$

The first term is $\mathcal{C}^\infty$ and $\overline{x} \notin \operatorname{supp}(P(D)((1-\varphi)u))$, so $\overline{x} \notin \operatorname{sing\,supp}(P(D)u)$. □

It remains to show the converse of (6.26) where $P(D)$ is assumed to be hypoelliptic. Take F, a parametrix for $P(D)$ with $\operatorname{sing\,supp} u \subset \{0\}$ and assume, or rather arrange, that F have compact support. In fact if $\overline{x} \notin \operatorname{sing\,supp}(P(D)u)$ we can arrange that

$$(\operatorname{supp}(F) + \overline{x}) \cap \operatorname{sing\,supp}(P(D)u) = \phi\,.$$

Now $P(D)F = \delta\psi$ with $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so

$$u = \delta * u = (P(D)F) * u - \psi * u.$$

Since $\psi * u \in \mathcal{C}^\infty$ it suffices to show that $\bar{x} \notin \operatorname{sing\,supp}\left((P(D)u) * f\right)$.

Take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\varphi f \in \mathcal{C}^\infty$, $f = P(D)u$ but

$$(\operatorname{supp} F + \overline{x}) \cap \operatorname{supp}(\varphi) = 0\,.$$

Then $f = f_1 + f_2$, $f_1 = \varphi f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so

$$f * F = f_1 * F + f_2 * F$$

where $f_1 * F \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\overline{x} \notin \operatorname{supp}(f_2 * F)$. It follows that $\overline{x} \notin \operatorname{sing\,supp}(u)$.

EXAMPLE 6.1. If u is holomorphic on $\mathbb{R}^n$, $\overline{\partial} u = 0$, then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Recall from last time that a differential operator $P(D)$ is said to be hypoelliptic if there exists $F \in \mathcal{S}'(\mathbb{R}^n)$ with

(6.27) $$P(D)F - \delta \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ and } \operatorname{sing\,supp}(F) \subset \{0\}\,.$$

The second condition here means that if $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\varphi(x) = 1$ in $|x| < \epsilon$ for some $\epsilon > 0$ then $(1-\varphi)F \in \mathcal{C}^\infty(\mathbb{R}^n)$. Since $P(D)((1-\varphi)F) \in \mathcal{C}^\infty(\mathbb{R}^n)$ we conclude that

$$P(D)(\varphi F) - \delta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$$

and we may well suppose that F, replaced now by φF, has compact support. Last time I showed that

$$\text{If } P(D) \text{ is hypoelliptic and } u \in \mathcal{S}'(\mathbb{R}^n) \text{ then}$$
$$\operatorname{sing\,supp}(u) = \operatorname{sing\,supp}(P(D)u)\,.$$

I will remind you of the proof later.

First however I want to discuss the important notion of *ellipticity*. Remember that $P(D)$ is 'really' just a polynomial, called the *characteristic polynomial*

$$P(\xi) = \sum_{|\alpha| \le m} C_\alpha \xi^\alpha \,.$$

It has the property

$$\widehat{P(D)u}(\xi) = P(\xi)\hat{u}(\xi) \ \forall \ u \in \mathcal{S}'(\mathbb{R}^n) \,.$$

This shows (if it isn't already obvious) that we can remove $P(\xi)$ from $P(D)$ thought of as an operator on $\mathcal{S}'(\mathbb{R}^n)$.

We can think of *inverting* $P(D)$ by dividing by $P(\xi)$. This works well provided $P(\xi) \neq 0$, for all $\xi \in \mathbb{R}^n$. An example of this is

$$P(\xi) = |\xi|^2 + 1 = \sum_{j=1}^n +1 \,.$$

However even the Laplacian, $\Delta = \sum_{j=1}^n D_j^2$, does not satisfy this rather stringent condition.

It is reasonable to expect the top order derivatives to be the most important. We therefore consider

$$P_m(\xi) = \sum_{|\alpha| = m} C_\alpha \xi^\alpha$$

the leading part, or *principal symbol*, of $P(D)$.

Definition 6.11. *A polynomial $P(\xi)$, or $P(D)$, is said to be* elliptic *of order m provided $P_m(\xi) \neq 0$ for all $0 \neq \xi \in \mathbb{R}^n$.*

So what I want to show today is

Theorem 6.12. *Every elliptic differential operator $P(D)$ is hypoelliptic.*

We want to find a *parametrix* for $P(D)$; we already know that we might as well suppose that F has compact support. Taking the Fourier transform of (6.27) we see that $\widehat{F}$ should satisfy

$$P(\xi)\widehat{F}(\xi) = 1 + \widehat{\psi}, \ \widehat{\psi} \in \mathcal{S}(\mathbb{R}^n) \,. \tag{6.28}$$

Here we use the fact that $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, so $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^n)$ too.

First suppose that $P(\xi) = P_m(\xi)$ is actually homogeneous of degree m. Thus

$$P_m(\xi) = |\xi|^m P_m(\widehat{\xi}), \ \widehat{\xi} = \xi / |\xi| \ , \xi \neq 0 \,.$$

The assumption at ellipticity means that

$$P_m(\widehat{\xi}) \neq 0 \;\forall\; \widehat{\xi} \in \mathcal{S}^{n-1} = \{\xi \in \mathbb{R}^n; |\xi| = 1\}\ . \tag{6.29}$$

Since $\mathcal{S}^{n-1}$ is *compact* and P_m is continuous

$$\left|P_m(\widehat{\xi})\right| \geq C > 0 \;\forall\; \widehat{\xi} \in \mathcal{S}^{n-1}\,, \tag{6.30}$$

for some constant C. Using homogeneity

$$\left|P_m(\widehat{\xi})\right| \geq C\,|\xi|^m\ ,\ C > 0 \;\forall\; \xi \in \mathbb{R}^n\,. \tag{6.31}$$

Now, to get $\widehat{F}$ from (6.28) we want to divide by $P_m(\xi)$ or multiply by $1/P_m(\xi)$. The only problem with defining $1/P_m(\xi)$ is at $\xi = 0$. We shall simply avoid this unfortunate point by choosing $P \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ as before, with $\varphi(\xi) = 1$ in $|\xi| \leq 1$.

Lemma 6.13. *If $P_m(\xi)$ is homogeneous of degree m and elliptic then*

$$Q(\xi) = \frac{(1-\varphi(\xi))}{P_m(\xi)} \in \mathcal{S}'(\mathbb{R}^n) \tag{6.32}$$

is the Fourier transform of a parametrix for $P_m(D)$, satisfying (6.27).

Proof. Clearly $Q(\xi)$ is a continuous function and $|Q(\xi)| \leq C(1+|\xi|)^{-m} \;\forall\; \xi \in \mathbb{R}^n$, so $Q \in \mathcal{S}'(\mathbb{R}^n)$. It therefore *is* the Fourier transform of some $F \in \mathcal{S}'(\mathbb{R}^n)$. Furthermore

$$\begin{aligned}\widehat{P_m(D)}F(\xi) &= P_m(\xi)\widehat{F} = P_m(\xi)Q(\xi)\\ &= 1 - \varphi(\xi)\,,\\ \Rightarrow P_m(D)F &= \delta + \psi\,,\ \widehat{\psi}(\xi) = -\varphi(\xi)\,.\end{aligned}$$

Since $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$. Thus F *is* a parametrix for $P_m(D)$. We still need to show the 'hard part' that

$$\operatorname{sing\,supp}(F) \subset \{0\}\ . \tag{6.33}$$

□

We can show (6.33) by considering the distributions $x^\alpha F$. The idea is that for $|\alpha|$ large, x^α vanishes rather rapidly at the origin and this should 'weaken' the singularity of F there. In fact we shall show that

$$x^\alpha F \in H^{|\alpha|+m-n-1}(\mathbb{R}^n)\,,\ |\alpha| > n+1-m\,. \tag{6.34}$$

If you recall, these Sobolev spaces are defined in terms of the Fourier transform, namely we must show that

$$\widehat{x^\alpha F} \in \langle\xi\rangle^{-|\alpha|-m+n+1}L^2(\mathbb{R}^n)\,.$$

Now $\widehat{x^\alpha F} = (-1)^{|\alpha|} D^\alpha{}_\xi \widehat{F}$, so what we need to cinsider is the behaviour of the derivatives of $\widehat{F}$, which is just $Q(\xi)$ in (6.32).

Lemma 6.14. *Let $P(\xi)$ be a polynomial of degree m satisfying*

$$|P(\xi)| \geq C\,|\xi|^m \ \text{ in } \ |\xi| > 1/C \text{ for some } C > 0\,, \tag{6.35}$$

then for some constants C_α

$$\left| D^\alpha \frac{1}{P(\xi)} \right| \leq C_\alpha\,|\xi|^{-m-|\alpha|} \ \text{ in } \ |\xi| > 1/C\,. \tag{6.36}$$

Proof. The estimate in (6.36) for $\alpha = 0$ is just (6.35). To prove the higher estimates that for each α there is a polynomial of degree at most $(m-1)\,|\alpha|$ such that

$$D^\alpha \frac{1}{P(\xi)} = \frac{L_\alpha(\xi)}{(P(\xi))^{1+|\alpha|}}\,. \tag{6.37}$$

Once we know (6.37) we get (6.36) straight away since

$$\left| D^\alpha \frac{1}{P(\xi)} \right| \leq \frac{C'_\alpha\,|\xi|^{(m-1)|\alpha|}}{C^{1+|\alpha|}\,|\xi|^{m(1+|\alpha|)}} \leq C_\alpha\,|\xi|^{-m-|\alpha|}\,.$$

We can prove (6.37) by induction, since it is certainly true for $\alpha = 0$. Suppose it is true for $|\alpha| \leq k$. To get the same identity for each β with $|\beta| = k+1$ it is enough to differentiate one of the identities with $|\alpha| = k$ once. Thus

$$D^\beta \frac{1}{P(\xi)} = D_j D^\alpha \frac{1}{P(\xi)} = \frac{D_j L_\alpha(\xi)}{P(\xi)^{1+|\alpha|}} - \frac{(1+|\alpha|) L_\alpha D_j P(\xi)}{(P(\xi))^{2+|\alpha|}}\,.$$

Since $L_\beta(\xi) = P(\xi) D_j L_\alpha(\xi) - (1+|\alpha|) L_\alpha(\xi) D_j P(\xi)$ is a polynomial of degree at most $(m-1)\,|\alpha| + m - 1 = (m-1)\,|\beta|$ this proves the lemma. $\square$

Going backwards, observe that $Q(\xi) = \frac{1-\varphi}{P_m(\xi)}$ is smooth in $|\xi| \leq 1/C$, so (6.36) implies that

$$|D^\alpha Q(\xi)| \leq C_\alpha (1 + |\xi|)^{-m-|\alpha|} \tag{6.38}$$

$$\Rightarrow \langle\xi\rangle^\ell D^\alpha Q \in L^2(\mathbb{R}^n) \text{ if } \ell - m - |\alpha| < -\frac{n}{2}\,,$$

which certainly holds if $\ell = |\alpha| + m - n - 1$, giving (6.34). Now, by Sobolev's embedding theorem

$$x^\alpha F \in \mathcal{C}^k \text{ if } |\alpha| > n + 1 - m + k + \frac{n}{2}\,.$$

In particular this means that if we choose $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}(\mu)$ then for every k, $\mu/|x|^{2k}$ is smooth and

$$\mu F = \frac{\mu}{|x|^{2k}} |x|^{2k} F \in \mathcal{C}^{2\ell-2n}\,,\ \ell > n\,.$$

Thus $\mu F \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and this is what we wanted to show, $\operatorname{sing\,supp}(F) \subset \{0\}$.

So now we have actually proved that $P_m(D)$ is hypoelliptic if it is elliptic. Rather than go through the proof again to make sure, let me go on to the general case and in doing so review it.

Proof. *Proof of theorem.* We need to show that if $P(\xi)$ is elliptic then $P(D)$ has a parametrix F as in (6.27). From the discussion above the ellipticity of $P(\xi)$ implies (and is equivalent to)

$$|P_m(\xi)| \geq c\,|\xi|^m\ ,\ c > 0\,.$$

On the other hand

$$P(\xi) - P_m(\xi) = \sum_{|\alpha|<m} C_\alpha \xi^\alpha$$

is a polynomial of degree at most $m-1$, so

$$|P(\xi) - P_m(\xi)|\,2 \leq C'(1+|\xi|)^{m-1}\,.$$

This means that id $C > 0$ is large enough then in $|\xi| > C$, $C'(1 + |\xi|)^{m-1} < \frac{c}{2}\,|\xi|^m$, so

$$\begin{aligned} |P(\xi)| &\geq |P_m(\xi)| - |P(\xi) - P_m(\xi)| \\ &\geq c\,|\xi|^m - C'(1+|\xi|)^{m-1} \geq \frac{c}{2}\,|\xi|^m\,. \end{aligned}$$

This means that $P(\xi)$ itself satisfies the conditions of Lemma 6.14. Thus if $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 in a large enough ball then $Q(xi) = (1-\varphi(\xi))/P(\xi)$ in $\mathcal{C}^\infty$ and satisfies (6.36) which can be written

$$|D^\alpha Q(\xi)| \leq C_\alpha(1+|\xi|)^{m-|\alpha|}\,.$$

The discussion above now shows that defining $F \in \mathcal{S}'(\mathbb{R}^n)$ by $\widehat{F}(\xi) = Q(\xi)$ gives a solution to (6.27).

□

The last step in the proof is to show that if $F \in \mathcal{S}'(\mathbb{R}^n)$ has compact support, and satisfies (6.27), then

$$\begin{aligned} &u \in \mathcal{S}(\mathbb{R}^n)\,,\ P(D)u \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n) \\ &\Rightarrow u = F * (P(D)u) - \psi * u \in \mathcal{C}^\infty(\mathbb{R}^n)\,. \end{aligned}$$

Let me refine this result a little bit.

PROPOSITION 6.15. *If $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\mu \in \mathcal{S}'(\mathbb{R}^n)$ has compact support then*

$$\operatorname{sing\,supp}(u * f) \subset \operatorname{sing\,supp}(u) + \operatorname{sing\,supp}(f).$$

PROOF. We need to show that $p \notin \operatorname{sing\,supp}(u) \in \operatorname{sing\,supp}(f)$ then $p \notin \operatorname{sing\,supp}(u * f)$. Once we can fix p, we might as well suppose that f has compact support too. Indeed, choose a large ball $B(R, 0)$ so that

$$z \notin B(0, R) \Rightarrow p \notin \operatorname{supp}(u) + B(0, R)\,.$$

This is possible by the assumed boundedness of $\operatorname{supp}(u)$. Then choose $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\varphi = 1$ on $B(0, R)$; it follows from Theorem L16.2, or rather its extension to distributions, that $\phi \notin \operatorname{supp}(u(1 - \varphi)f)$, so we can replace f by φf, noting that $\operatorname{sing\,supp}(\varphi f) \subset \operatorname{sing\,supp}(f)$. Now if f has compact support we can choose compact neighbourhoods K_1, K_2 of $\operatorname{sing\,supp}(u)$ and $\operatorname{sing\,supp}(f)$ such that $p \notin K_1 + K_2$. Furthermore we an decompose $u = u_1 + u_2$, $f = f_1 + f_2$ so that $\operatorname{supp}(u_1) \subset K_1$, $\operatorname{supp}(f_2) \subset K_2$ and $u_2, f_2 \in \mathcal{C}^\infty(\mathbb{R}^n)$. It follows that

$$u * f = u_1 * f_1 + u_2 * f_2 + u_1 * f_2 + u_2 * f_2\,.$$

Now, $p \notin \operatorname{supp}(u_1 * f_1)$, by the support property of convolution and the three other terms are $\mathcal{C}^\infty$, since at least one of the factors is $\mathcal{C}^\infty$. Thus $p \notin \operatorname{sing\,supp}(u * f)$. □

The most important example of a differential operator which is hypoelliptic, but not elliptic, is the heat operator

$$\partial_t + \Delta = \partial_t - \sum_{j=1}^{n} \partial_{x_j}^2\,. \tag{6.39}$$

In fact the distribution

$$E(t, x) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) & t \geq 0 \\ 0 & t \leq 0 \end{cases} \tag{6.40}$$

is a fundamental solution. First we need to check that E *is* a distribution. Certainly E is $\mathcal{C}^\infty$ in $t > 0$. Moreover as $t \downarrow 0$ in $x \neq 0$ it vanishes with all derivatives, so it is $\mathcal{C}^\infty$ except at $t = 0$, $x = 0$. Since it is clearly measurable we will check that it is locally integrable near the origin, i.e.,

$$\int_{\substack{0 \leq t \leq 1 \\ |x| \leq 1}} E(t, x)\; dx\, dt < \infty\,, \tag{6.41}$$

since $E \geq 0$. We can change variables, setting $X = x/t^{1/2}$, so $dx = t^{n/2}\,dX$ and the integral becomes

$$\frac{1}{(4\pi)^{n/2}} \int_0^{t'} \int_{|X| \leq t^{-1/2}} \exp(-\frac{|X|^2}{4})\,dx\,dt \ < \infty\,.$$

Since E is actually bounded near infinity, it follows that $E \in \mathcal{S}'\mathbb{R}^n$,

$$E(\varphi) = \int_{t \geq 0} E(t,x)\varphi(t,x)\,dx\,dt \ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^{n+1})\,.$$

As before we want to compute

$$\begin{aligned}(\partial_t + \Delta)E(\varphi) &= E(-\partial_t\varphi + \Delta\varphi) \\ &= \lim_{\mathcal{E}\downarrow 0} \int_{\mathcal{E}}^{\infty} \int_{\mathbb{R}^n} E(t,x)(-\partial_t\varphi + \Delta\varphi)\,dx\,dt\,.\end{aligned} \tag{6.42}$$

First we check that $(\partial_t + \Delta)E = 0$ in $t > 0$, where it is a $\mathcal{C}^\infty$ function. This is a straightforward computation:

$$\begin{aligned}\partial_t E &= -\frac{n}{2t}E + \frac{|x|^2}{4t^2}E \\ \partial_{x_j}E &= -\frac{x_j}{2t}E\,,\ \partial^2_{x_j}E = -\frac{1}{2t}E + \frac{x_j^2}{4t^2}E \\ \Rightarrow \Delta E &= \frac{n}{2t}E + \frac{|x|^2}{4t^2}E\,.\end{aligned}$$

Now we can integrate by parts in (6.42) to get

$$(\partial_t + \Delta)E(\varphi) = \lim_{\mathcal{E}\downarrow 0} \int_{\mathbb{R}^n} \varphi(\mathcal{E},x)\frac{e^{-|x|^2/4\mathcal{E}}}{(4\pi\mathcal{E})^{n/2}}\,dx\,.$$

Making the same change of variables as before, $X = x/2\mathcal{E}^{1/2}$,

$$(\partial_t + \Delta)E(\varphi) = \lim_{\mathcal{E}\downarrow 0} \int_{\mathbb{R}^n} \varphi(\mathcal{E},\mathcal{E}^{1/2}X)\frac{e^{-|x|^2}}{\pi^{n/2}}\,dX\,.$$

As $\mathcal{E} \downarrow 0$ the integral here is bounded by the integrable function $C\exp(-|X|^2)$, for some $C > 0$, so by Lebesgue's theorem of dominated convergence, conveys to the integral of the limit. This is

$$\varphi(0,0)\cdot\int_{\mathbb{R}^n} e^{-|x|^2}\frac{dx}{\pi^{n/2}} = \varphi(0,0)\,.$$

Thus

$$(\partial_t + \Delta)E(\varphi) = \varphi(0,0) \Rightarrow (\partial_t + \Delta)E = \delta_t\delta_x\,,$$

so E is indeed a fundamental solution. Since it vanishes in $t < 0$ it is called a *forward fundamrntal* solution.

Let's see what we can use it for.

PROPOSITION 6.16. *If* $f \in \mathcal{S}'\mathbb{R}^n$ *has compact support* $\exists\,!u \in \mathcal{S}'\mathbb{R}^n$ *with* $\operatorname{supp}(m) \subset \{t \geq -T\}$ *for some* T *and*

$$(\partial_t + \Delta)u = f \text{ in } \mathbb{R}^{n+1}\,. \tag{6.43}$$

PROOF. Naturally we try $u = E * f$. That it satisfies (6.43)follows from the properties of convolution. Similarly if T is such that $\operatorname{supp}(f) \subset \{t \geq T\}$ then

$$\operatorname{supp}(u) \subset \operatorname{supp}(f) + \operatorname{supp}(E) \subset \{t \geq T]\ .$$

So we need to show *uniqueness*. If $u_1, u_2 \in \mathcal{S}'\mathbb{R}^n$ in two solutions of (6.43) then their difference $v = u_1 - u_2$ satisfies the 'homogeneous' equation $(\partial_t + \Delta)v = 0$. Furthermore, $v = 0$ in $t < T'$ for some T'. Given any $E \in \mathbb{R}$ choose $\varphi(t) \in \mathcal{C}^\infty(\mathbb{R})$ with $\varphi(t) = 0$ in $t > \bar{t} + 1$, $\varphi(t) = 1$ in $t < \bar{t}$ and consider

$$E_{\bar{t}} = \varphi(t)E = F_1 + F_2\,,$$

where $F_1 = \psi E_{\bar{t}}$ for some $\psi \in \mathcal{C}^\infty_c\mathbb{R}^{n+1})$, $\psi = 1$ near 0. Thus F_1 has comapct support and in fact $F_2 \in \mathcal{S}\mathbb{R}^n$. I ask you to check this last statement as Problem L18.P1.

Anyway,

$$(\partial_t + \Delta)(F_1 + F_2) = \delta + \psi \in \mathcal{S}\mathbb{R}^n\,,\ \psi_{\bar{t}} = 0\ t \leq \bar{t}\,.$$

Now,

$$(\partial_t + \Delta)(E_t * u) = 0 = u + \psi_{\bar{t}} * u\,.$$

Since $\operatorname{supp}(\psi_{\bar{t}}) \subset \left\{t \geq \bar{t}\right]$, the second tier here is supported in $t \geq \bar{t} \geq T'$. Thus $u = 0$ in $t < \bar{t} + T'$, but $\bar{t}$ is arbitrary, so $u = 0$. □

Notice that the assumption that $u \in \mathcal{S}'\mathbb{R}^n$ is not redundant in the statement of the Proposition, if we allow "large" solutions they become non-unique. Problem L18.P2 asks you to apply the fundamental solution to solve the initial value problem for the heat operator.

Next we make similar use of the fundamental solution for Laplace's operator. If $n \geq 3$ the

$$E = C_n\,|x|^{-n+2} \tag{6.44}$$

is a fundamental solution. You should check that $\Delta E_n = 0$ in $x \neq 0$ directly, I will show later that $\Delta E_n = \delta$, for the appropriate choice of C_n, but you can do it directly, as in the case $n = 3$.

THEOREM 6.17. *If* $f \in \mathcal{S}\mathbb{R}^n$ $\exists\ !u \in \mathcal{C}^\infty_0\mathbb{R}^n$ *such that* $\Delta u = f$.

Proof. Since convolution $u = E * f \in \mathcal{S}'\mathbb{R}^n \cap \mathcal{C}^\infty\mathbb{R}^n$ is defined we certainly get a solution to $\Delta u = f$ this way. We need to check that $u \in \mathcal{C}_0^\infty\mathbb{R}^n$. First we know that Δ is hypoelliptic so we can decompose

$$E = F_1 + F_2\,,\ \ F_1 \in \mathcal{S}'\mathbb{R}^n\,,\, \operatorname{supp} F, \Subset \mathbb{R}^n$$

and then $F_2 \in \mathcal{C}^\infty\mathbb{R}^n$. In fact we can see from (6.44) that

$$|D^\alpha F_2(x)| \le C_\alpha(1+|x|)^{-n+2-|\alpha|}\,.$$

Now, $F_1 * f \in \mathcal{S}\mathbb{R}^n$, as we showed before, and continuing the integral we see that

$$\begin{aligned}|D^\alpha u| &\le |D^\alpha F_2 * f| + C_N(1+|x|)^{-N}\ \forall\ N\\ &\le C'_\alpha(1+|x|)^{-n+2-|\alpha|}\,.\end{aligned}$$

Since $n > 2$ it follows that $u \in \mathcal{C}_0^\infty\mathbb{R}^n$.

So only the uniqueness remains. If there are two solutions, u_1, u_2 for a given f then $v = u_1 - u_2 \in \mathcal{C}_0^\infty\mathbb{R}^n$ satisfies $\Delta v = 0$. Since $v \in \mathcal{S}'\mathbb{R}^n$ we can take the Fourier transform and see that

$$|\chi|^2\,\widehat{v}(\chi) = 0 \Rightarrow \operatorname{supp}(\widehat{v}) \subset \{0\}\ .$$

an earlier problem was to conclude from this that $\widehat{v} = \sum_{|\alpha|\le m} C_\alpha D^\alpha\delta$ for some constants C_α. This in turn implies that v is a polynomial. However the only polynomials in $\mathcal{C}_0^0\mathbb{R}^n$ are identically 0. Thus $v = 0$ and uniqueness follows. □

7. Cone support and wavefront set

In discussing the singular support of a tempered distibution above, notice that

$$\operatorname{singsupp}(u) = \emptyset$$

only implies that $u \in \mathcal{C}^\infty(\mathbb{R}^n)$, not as one might want, that $u \in \mathcal{S}(\mathbb{R}^n)$. We can however 'refine' the concept of singular support a little to get this.

Let us think of the sphere $\mathbb{S}^{n-1}$ as the set of 'asymptotic directions' in $\mathbb{R}^n$. That is, we identify a point in $\mathbb{S}^{n-1}$ with a half-line $\{a\bar{x}; a \in (0,\infty)\}$ for $0 \ne \bar{x} \in \mathbb{R}^n$. Since two points give the same half-line if and only if they are positive multiples of each other, this means we think of the sphere as the quotient

$$\mathbb{S}^{n-1} = (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+. \tag{7.1}$$

Of course if we have a metric on $\mathbb{R}^n$, for instance the usual Euclidean metric, then we can identify $\mathbb{S}^{n-1}$ with the unit sphere. However (7.1) does not require a choice of metric.

Now, suppose we consider functions on $\mathbb{R}^n \setminus \{0\}$ which are (positively) homogeneous of degree 0. That is $f(a\bar{x}) = f(\bar{x})$, for all $a > 0$, and they are just functions on $\mathbb{S}^{n-1}$. Smooth functions on $\mathbb{S}^{n-1}$ correspond (if you like by definition) with smooth functions on $\mathbb{R}^n \setminus \{0\}$ which are homogeneous of degree 0. Let us take such a function $\psi \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$, $\psi(ax) = \psi(x)$ for all $a > 0$. Now, to make this smooth on $\mathbb{R}^n$ we need to cut it off near 0. So choose a cutoff function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, with $\chi(x) = 1$ in $|x| < 1$. Then

$$\psi_R(x) = \psi(x)(1 - \chi(x/R)) \in \mathcal{C}^\infty(\mathbb{R}^n), \tag{7.2}$$

for any $R > 0$. This function is supported in $|x| \ge R$. Now, if ψ has support near some point $\omega \in \mathbb{S}^{n-1}$ then for R large the corresponding function ψ_R will 'localize near ω as a point at infinity of $\mathbb{R}^n$.' Rather than try to understand this directly, let us consider a corresponding analytic construction.

First of all, a function of the form ψ_R is a multiplier on $\mathcal{S}(\mathbb{R}^n)$. That is,

$$\psi_R \cdot : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{7.3}$$

To see this, the main problem is to estimate the derivatives at infinity, since the product of smooth functions is smooth. This in turn amounts to estimating the deriviatives of ψ in $|x| \ge 1$. This we can do using the homogeneity.

LEMMA 7.1. *If* $\psi \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ *is homogeneous of degree* 0 *then*

$$|D^\alpha \psi| \le C_\alpha |x|^{-|\alpha|}. \tag{7.4}$$

PROOF. I should not have even called this a lemma. By the chain rule, the derivative of order α is a homogeneous function of degree $-|\alpha|$ from which (7.4) follows. □

For the smoothed versio, ψ_R, of ψ this gives the estimates

$$|D^\alpha \psi_R(x)| \le C_\alpha \langle x \rangle^{-|\alpha|}. \tag{7.5}$$

This allows us to estimate the derivatives of the product of a Schwartz function and ψ_R :

$$\begin{aligned} &x^\beta D^\alpha(\psi_R f) \\ &= \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} D^{\alpha-\gamma}\psi_R x^\beta D^\gamma f \Longrightarrow \sup_{|x|\ge 1} |x^\beta D^\alpha(\psi_R f)| \le C \sup \|f\|_k \end{aligned} \tag{7.6}$$

for some seminorm on $\mathcal{S}(\mathbb{R}^n)$. Thus the map (7.3) is actually continuous. This continuity means that ψ_R is a multiplier on $\mathcal{S}'(\mathbb{R}^n)$, defined as usual by duality:

$$\psi_R u(f) = u(\psi_R f) \ \forall\ f \in \mathcal{S}(\mathbb{R}^n). \tag{7.7}$$

DEFINITION 7.2. *The cone-support and cone-singular-support of a tempered distribution are the subsets* $\mathrm{Csp}(u) \subset \mathbb{R}^n \cup \mathbb{S}^{n-1}$ *and* $\mathrm{Css}(u) \subset \mathbb{R}^n \cup \mathbb{S}^{n-1}$ *defined by the conditions*

$$\begin{aligned} \mathrm{Csp}(u) \cap \mathbb{R}^n &= \mathrm{supp}(u) \\ (\mathrm{Csp}(u))^{\complement} \cap \mathbb{S}^{n-1} &= \{\omega \in \mathbb{S}^{n-1}; \\ &\qquad \exists\ R > 0,\ \psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1}),\ \psi(\omega) \neq 0,\ \psi_R u = 0\}, \\ \mathrm{Css}(u) \cap \mathbb{R}^n &= \mathrm{singsupp}(u) \\ (\mathrm{Css}(u))^{\complement} \cap \mathbb{S}^{n-1} &= \{\omega \in \mathbb{S}^{n-1}; \\ &\qquad \exists\ R > 0,\ \psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1}),\ \psi(\omega) \neq 0,\ \psi_R u \in \mathcal{S}(\mathbb{R}^n)\}. \end{aligned} \tag{7.8}$$

That is, on the $\mathbb{R}^n$ part these are the same sets as before but 'at infinity' they are defined by conic localization on $\mathbb{S}^{n-1}$.

In considering $\mathrm{Csp}(u)$ and $\mathrm{Css}(u)$ it is convenient to combine $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ into a compactification of $\mathbb{R}^n$. To do so (topologically) let us identify $\mathbb{R}^n$ with the interior of the unit ball with respect to the Euclidean metric using the map

$$\mathbb{R}^n \ni x \longmapsto \frac{x}{\langle x\rangle} \in \{y \in \mathbb{R}^n; |y| \le 1\} = \mathbb{B}^n. \tag{7.9}$$

Clearly $|x| < \langle x\rangle$ and for $0 \le a < 1$, $|x| = a\langle x\rangle$ has only the solution $|x| = a/(1-a^2)^{\frac{1}{2}}$. Thus if we combine (7.9) with the identification of $\mathbb{S}^n$ with the unit sphere we get an identification

$$\mathbb{R}^n \cup \mathbb{S}^{n-1} \simeq \mathbb{B}^n. \tag{7.10}$$

Using this identification we can, and will, regard $\mathrm{Csp}(u)$ and $\mathrm{Css}(u)$ as subsets of $\mathbb{B}^n$.[12]

LEMMA 7.3. *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$, $\mathrm{Csp}(u)$ *and* $\mathrm{Css}(u)$ *are closed subsets of* $\mathbb{B}^n$ *and if* $\tilde\psi \in \mathcal{C}^\infty(\mathbb{S}^n)$ *has* $\mathrm{supp}(\tilde\psi) \cap \mathrm{Css}(u) = \emptyset$ *then for* R *sufficiently large* $\tilde\psi_R u \in \mathcal{S}(\mathbb{R}^n)$.

PROOF. Directly from the definition we know that $\mathrm{Csp}(u) \cap \mathbb{R}^n$ is closed, as is $\mathrm{Css}(u) \cap \mathbb{R}^n$. Thus, in each case, we need to show that if $\omega \in \mathbb{S}^{n-1}$ and $\omega \notin \mathrm{Csp}(u)$ then $\mathrm{Csp}(u)$ is disjoint from some neighbourhood of ω in $\mathbb{B}^n$. However, by definition,

$$U = \{x \in \mathbb{R}^n; \psi_R(x) \neq 0\} \cup \{\omega' \in \mathbb{S}^{n-1}; \psi(\omega') \neq 0\}$$

[12] In fact while the topology here is correct the smooth structure on $\mathbb{B}^n$ is *not the right one*™– see Problem?? For our purposes here this issue is irrelevant.

is such a neighbourhood. Thus the fact that $\mathrm{Csp}(u)$ is closed follows directly from the definition. The argument for $\mathrm{Css}(u)$ is essentially the same.

The second result follows by the use of a partition of unity on $\mathbb{S}^{n-1}$. Thus, for each point in $\mathrm{supp}(\psi) \subset \mathbb{S}^{n-1}$ there exists a conic localizer for which $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$. By compactness we may choose a finite number of these functions ψ_j such that the open sets $\{\psi_j(\omega) > 0\}$ cover $\mathrm{supp}(\tilde{\psi})$. By assumption $(\psi_j)_{R_j} u \in \mathcal{S}(\mathbb{R}^n)$ for some $R_j > 0$. However this will remain true if R_j is increased, so we may suppose that $R_j = R$ is independent of j. Then for function

$$\mu = \sum_j |\psi_j|^2 \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$$

we have $\mu_R u \in \mathcal{S}(\mathbb{R}^n)$. Since $\tilde{\psi} = \psi' \mu$ for some $\mu \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ it follows that $\tilde{\psi}_{R+1} u \in \mathcal{S}(\mathbb{R}^n)$ as claimed. $\square$

Corollary 7.4. *If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *then* $\mathrm{Css}(u) = \emptyset$ *if and only if* $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Certainly $\mathrm{Css}(u) = \emptyset$ if $u \in \mathcal{S}(\mathbb{R}^n)$. If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathrm{Css}(u) = \emptyset$ then from Lemma 7.3, $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$ where $\psi = 1$. Thus $v = (1 - \psi_R)u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ has $\mathrm{singsupp}(v) = \emptyset$ so $v \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and hence $u \in \mathcal{S}(\mathbb{R}^n)$. $\square$

Of course the analogous result for $\mathrm{Csp}(u)$, that $\mathrm{Csp}(u) = \emptyset$ if and only if $u = 0$ follows from the fact that this is true if $\mathrm{supp}(u) = \emptyset$. I will treat a few other properties as self-evident. For instance

$$\mathrm{Csp}(\phi u) \subset \mathrm{Csp}(u),\ \mathrm{Css}(\phi u) \subset \mathrm{Css}(u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n),\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{7.11}$$

and

$$\begin{aligned} \mathrm{Csp}(c_1u_1 + c_2u_2) &\subset \mathrm{Csp}(u_1) \cup \mathrm{Csp}(u_2), \\ \mathrm{Css}(c_1u_1 + c_2u_2) &\subset \mathrm{Css}(u_1) \cup \mathrm{Css}(u_2) \\ &\qquad \forall\ u_1, u_2 \in \mathcal{S}'(\mathbb{R}^n),\ c_1, c_2 \in \mathbb{C}. \end{aligned} \tag{7.12}$$

One useful consequence of having the cone support at our disposal is that we can discuss sufficient conditions to allow us to multiply distributions; we will get better conditions below using the same idea but applied to the wavefront set but this preliminary discussion is used there. In general the product of two distributions is not defined, and indeed not definable, as a distribution. However, we can always multiply an element of $\mathcal{S}'(\mathbb{R}^n)$ and an element of $\mathcal{S}(\mathbb{R}^n)$.

To try to understand multiplication look at the question of *pairing* between two distributions.

LEMMA 7.5. *If $K_i \subset \mathbb{B}^n$, $i = 1, 2$, are two disjoint closed (hence compact) subsets then we can define an unambiguous pairing*

$$\begin{aligned}\{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{Css}(u) \subset K_1\} \times \{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{Css}(u) \subset K_2\} \ni (u_1, u_2) \\ \longrightarrow u_1(u_2) \in \mathbb{C}.\end{aligned} \tag{7.13}$$

PROOF. To define the pairing, choose a function $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ which is identically equal to 1 in a neighbourhood of $K_1 \cap \mathbb{S}^{n-1}$ and with support disjoint from $K_2 \cap \mathbb{S}^{n-1}$. Then extend it to be homogeneous, as above, and cut off to get ψ_R. If R is large enough $\operatorname{Csp}(\psi_R)$ is disjoint from K_2. Then $\psi_R + (1 - \psi)_R = 1 + \nu$ where $\nu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. We can find another function $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\psi_1 = \psi_R + \mu = 1$ in a neighbourhood of K_1 and with $\operatorname{Csp}(\psi_1)$ disjoint from K_2. Once we have this, for u_1 and u_2 as in (7.13),

$$\psi_1 u_2 \in \mathcal{S}(\mathbb{R}^n) \text{ and } (1 - \psi_1)u_1 \in \mathcal{S}(\mathbb{R}^n) \tag{7.14}$$

since in both cases Css is empty from the definition. Thus we can define the desired pairing between u_1 and u_2 by

$$u_1(u_2) = u_1(\psi_1 u_2) + u_2((1 - \psi_1)u_1). \tag{7.15}$$

Of course we should check that this definition is independent of the cut-off function used in it. However, if we go through the definition and choose a different function ψ' to start with, extend it homogeneoulsy and cut off (probably at a different R) and then find a correction term μ' then the 1-parameter linear homotopy between them

$$\psi_1(t) = t\psi_1 + (1 - t)\psi_1', \ t \in [0, 1] \tag{7.16}$$

satisfies all the conditions required of ψ_1 in formula (7.14). Thus in fact we get a smooth family of pairings, which we can write for the moment as

$$(u_1, u_2)_t = u_1(\psi_1(t)u_2) + u_2((1 - \psi_1(t))u_1). \tag{7.17}$$

By inspection, this is an affine-linear function of t with derivative

$$u_1((\psi_1 - \psi_1')u_2) + u_2((\psi_1' - \psi_1))u_1). \tag{7.18}$$

Now, we just have to justify moving the smooth function in (7.18) to see that this gives zero. This should be possible since $\operatorname{Csp}(\psi_1' - \psi_1)$ is disjoint from *both* K_1 and K_2.

In fact, to be very careful for once, we should construct another function χ in the same way as we constructed ψ_1 to be homogenous near infinity and smooth and such that $\mathrm{Csp}(\chi)$ is also disjoint from both K_1 and K_2 but $\chi = 1$ on $\mathrm{Csp}(\psi_1' - \psi_1)$. Then $\chi(\psi_1' - \psi_1) = \psi_1' - \psi_1$ so we can insert it in (7.18) and justify

$$
\begin{aligned}
(7.19)\quad u_1((\psi_1 - \psi_1')u_2) = u_1(\chi^2(\psi_1 - \psi_1')u_2) &= (\chi u_1)((\psi_1 - \psi_1')\chi u_2) \\
&= (\chi u_2)(\psi_1 - \psi_1')\chi u_1) = u_2(\psi_1 - \psi_1')\chi u_1).
\end{aligned}
$$

Here the second equality is just the identity for χ as a (multiplicative) linear map on $\mathcal{S}(\mathbb{R}^n)$ and hence $\mathcal{S}'(\mathbb{R}^n)$ and the operation to give the crucial, third, equality is permissible because both elements are in $\mathcal{S}(\mathbb{R}^n)$. □

Once we have defined the pairing between tempered distibutions with disjoint conic singular supports, in the sense of (7.14), (7.15), we can define the product under the same conditions. Namely to define the product of say u_1 and u_2 we simply set

$$
(7.20)\quad u_1u_2(\phi) = u_1(\phi u_2) = u_2(\phi u_1)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n), \quad \text{provided } \mathrm{Css}(u_1) \cap \mathrm{Css}(u_2) = \emptyset.
$$

Indeed, this would be true if one of u_1 or u_2 was itself in $\mathcal{S}(\mathbb{R}^n)$ and makes sense in general. I leave it to you to check the continuity statement required to prove that the product is actually a tempered distibution (Problem 78).

One can also give a similar discussion of the convolution of two tempered distributions. Once again we do not have a definition of $u * v$ as a tempered distribution for all u, $v \in \mathcal{S}'(\mathbb{R}^n)$. We do know how to define the convolution if either u or v is compactly supported, or if either is in $\mathcal{S}(\mathbb{R}^n)$. This leads directly to

Lemma 7.6. *If* $\mathrm{Css}(u) \cap \mathbb{S}^{n-1} = \emptyset$ *then* $u * v$ *is defined unambiguously by*

$$
(7.21)\qquad u * v = u_1 * v + u_2 * v,\ u_1 = (1 - \chi(\frac{x}{r}))u,\ u_2 = u - u_1
$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ *has* $\chi(x) = 1$ *in* $|x| \le 1$ *and* R *is sufficiently large; there is a similar definition if* $\mathrm{Css}(v) \cap \mathbb{S}^{n-1} = \emptyset$.

Proof. Since $\mathrm{Css}(u) \cap \mathbb{S}^{n-1} = \emptyset$, we know that $\mathrm{Css}(u_1) = \emptyset$ if R is large enough, so then both terms on the right in (7.21) are well-defined. To see that the result is independent of R just observe that the difference of the right-hand side for two values of R is of the form $w * v - w * v$ with w compactly supported. □

Now, we can go even further using a slightly more sophisticated decomposition based on

LEMMA 7.7. *If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\mathrm{Css}(u) \cap \Gamma = \emptyset$ *where* $\Gamma \subset \mathbb{S}^{n-1}$ *is a closed set, then* $u = u_1 + u_2$ *where* $\mathrm{Csp}(u_1) \cap \Gamma = \emptyset$ *and* $u_2 \in \mathcal{S}(\mathbb{R}^n)$*; in fact*

$$\begin{gathered} u = u_1' + u_1'' + u_2 \text{ where } u_1' \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n) \text{ and} \\ 0 \notin \mathrm{supp}(u_1''),\ x \in \mathbb{R}^n \setminus \{0\},\ x/|x| \in \Gamma \Longrightarrow x \notin \mathrm{supp}(u_1''). \end{gathered} \tag{7.22}$$

PROOF. A covering argument which you should provide. □

Let $\Gamma_i \subset \mathbb{R}^n$, $i = 1, 2$, be closed cones. That is they are closed sets such that if $x \in \Gamma_i$ and $a > 0$ then $ax \in \Gamma_i$. Suppose in addition that

$$\Gamma_1 \cap (-\Gamma_2) = \{0\}. \tag{7.23}$$

That is, if $x \in \Gamma_1$ and $-x \in \Gamma_2$ then $x = 0$. Then it follows that for some $c > 0$,

$$x \in \Gamma_1,\ y \in \Gamma_2 \Longrightarrow |x + y| \geq c(|x| + |y|). \tag{7.24}$$

To see this consider $x + y$ where $x \in \Gamma_1$, $y \in \Gamma_2$ and $|y| \leq |x|$. We can assume that $x \neq 0$, otherwise the estimate is trivially true with $c = 1$, and then $Y = y/|x| \in \Gamma_1$ and $X = x/|x| \in \Gamma_2$ have $|Y| \leq 1$ and $|X| = 1$. However $X + Y \neq 0$, since $|X| = 1$, so by the continuity of the sum, $|X + Y| \geq 2c > 0$ for some $c > 0$. Thus $|X + Y| \geq c(|X| + |Y|)$ and the result follows by scaling back. The other case, of $|x| \leq |y|$ follows by the same argument with x and y interchanged, so (7.24) is a consequence of (7.23).

LEMMA 7.8. *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathrm{Css}(\phi * u) \subset \mathrm{Css}(u) \cap \mathbb{S}^{n-1}. \tag{7.25}$$

PROOF. We already know that $\phi * u$ is smooth, so $\mathrm{Css}(\phi * u) \subset \mathbb{S}^{n-1}$. Thus, we need to show that if $\omega \in \mathbb{S}^{n-1}$ and $\omega \notin \mathrm{Css}(u)$ then $\omega \notin \mathrm{Css}(\phi * u)$.

Fix such a point $\omega \in \mathbb{S}^{n-1} \setminus \mathrm{Css}(u)$ and take a closed set $\Gamma \subset \mathbb{S}^{n-1}$ which is a neighbourhood of ω but which is still disjoint from $\mathrm{Css}(u)$ and then apply Lemma 7.7. The two terms $\phi * u_2$, where $u_2 \in \mathcal{S}(\mathbb{R}^n)$ and $\phi * u_1'$ where $u_1' \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ are both in $\mathcal{S}(\mathbb{R}^n)$ so we can assume that u has the support properties of u_1''. In particular there is a smaller closed subset $\Gamma_1 \subset \mathbb{S}^{n-1}$ which is still a neighbourhood of ω but which does not meet Γ_2, which is the closure of the complement of Γ. If we replace these Γ_i by the closed cones of which they are the 'cross-sections' then we are in the situation of (7.23) and (7.23), except for the signs. That is, there is a constant $c > 0$ such that

$$|x - y| \geq c(|x| + |y|). \tag{7.26}$$

Now, we can assume that there is a cutoff function ψ_R which has support in Γ_2 and is such that $u = \psi_R u$. For any conic cutoff, ψ'_R, with support in Γ_1

$$\psi'_R(\phi * u) = \langle \psi_R u, \phi(x - \cdot)\rangle = \langle u(y), \psi_R(y)\psi'_R(x)\phi(x-y)\rangle. \tag{7.27}$$

The continuity of u means that this is estimated by some Schwartz seminorm

$$\begin{aligned}\sup_{y,|\alpha|\le k} &|D_y^\alpha(\psi_R(y)\psi'_R(x)\phi(x-y))|(1+|y|)^k \\ &\le C_N\|\phi\| \sup_y (1+|x|+|y|)^{-N}(1+|y|)^k \le C_N\|\phi\|(1+|x|)^{-N+k}\end{aligned} \tag{7.28}$$

for some Schwartz seminorm on ϕ. Here we have used the estimate (7.24), in the form (7.26), using the properties of the supports of ψ'_R and ψ_R. Since this is true for any N and similar estimates hold for the derivatives, it follows that $\psi'_R(u * \phi) \in \mathcal{S}(\mathbb{R}^n)$ and hence that $\omega \notin \operatorname{Css}(u * \phi)$. □

Corollary 7.9. *Under the conditions of Lemma 7.6*

$$\operatorname{Css}(u * v) \subset (\operatorname{singsupp}(u) + \operatorname{singsupp}(v)) \cup (\operatorname{Css}(v) \cap \mathbb{S}^{n-1}). \tag{7.29}$$

Proof. We can apply Lemma 7.8 to the first term in (7.21) to conclude that it has conic singular support contained in the second term in (7.29). Thus it is enough to show that (7.29) holds when $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$. In that case we know that the singular support of the convolution is contained in the first term in (7.29), so it is enough to consider the conic singular support in the sphere at infinity. Thus, if $\omega \notin \operatorname{Css}(v)$ we need to show that $\omega \notin \operatorname{Css}(u * v)$. Using Lemma 7.7 we can decompose $v = v_1 + v_2 + v_3$ as a sum of a Schwartz term, a compact supported term and a term which does not have ω in its conic support. Then $u * v_1$ is Schwartz, $u * v_2$ has compact support and satisfies (7.29) and ω is not in the cone support of $u * v_3$. Thus (7.29) holds in general. □

Lemma 7.10. *If* $u,\ v \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\omega \in \operatorname{Css}(u) \cap \mathbb{S}^{n-1} \Longrightarrow -\omega \notin \operatorname{Css}(v)$ *then their convolution is defined unambiguously, using the pairing in Lemma 7.5, by*

$$u * v(\phi) = u(\check{v} * \phi) \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{7.30}$$

Proof. Since $\check{v}(x) = v(-x)$, $\operatorname{Css}(\check{v}) = -\operatorname{Css}(v)$ so applying Lemma 7.8 we know that

$$\operatorname{Css}(\check{v} * \phi) \subset -\operatorname{Css}(v) \cap \mathbb{S}^{n-1}. \tag{7.31}$$

Thus, $\mathrm{Css}(v) \cap \mathrm{Css}(\check{v} * \phi) = \emptyset$ and the pairing on the right in (7.30) is well-defined by Lemma 7.5. Continuity follows from your work in Problem 78. $\square$

In Problem 79 I ask you to get a bound on $\mathrm{Css}(u * v) \cap \mathbb{S}^{n-1}$ under the conditions in Lemma 7.10.

Let me do what is actually a fundamental computation.

Lemma 7.11. *For a conic cutoff, ψ_R, where $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$,*

$$\mathrm{Css}(\widehat{\psi_R}) \subset \{0\}. \tag{7.32}$$

Proof. This is actually much easier than it seems. Namely we already know that $D^\alpha(\psi_R)$ is smooth and homogeneous of degree $-|\alpha|$ near infinity. From the same argument it follows that

$$D^\alpha(x^\beta \psi_R) \in L^2(\mathbb{R}^n) \text{ if } |\alpha| > |\beta| + n/2 \tag{7.33}$$

since this is a smooth function homogeneous of degree less than $-n/2$ near infinity, hence square-integrable. Now, taking the Fourier transform gives

$$\xi^\alpha D^\beta(\widehat{\psi_R}) \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| > |\beta| + n/2. \tag{7.34}$$

If we localize in a cone near infinity, using a (completely unrelated) cutoff $\psi'_{R'}(\xi)$ then we must get a Schwartz function since

$$|\xi|^{|\alpha|}\psi'_{R'}(\xi) D^\beta(\widehat{\psi_R}) \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| > |\beta| + n/2 \Longrightarrow \psi'_{R'}(\xi)\widehat{\psi_R} \in \mathcal{S}(\mathbb{R}^n). \tag{7.35}$$

Indeed this argument applies anywhere that $\xi \neq 0$ and so shows that (7.32) holds. $\square$

Now, we have obtained some reasonable looking conditions under which the product uv or the convolution $u*v$ of two elements of $\mathcal{S}'(\mathbb{R}^n)$ is defined. However, reasonable as they might be there is clearly a flaw, or at least a deficiency, in the discussion. We know that in the simplest of cases,

$$\widehat{u * v} = \widehat{u}\widehat{v}. \tag{7.36}$$

Thus, it is very natural to expect a relationship between the conditions under which the product of the Fourier transforms is defined and the conditions under which the convolution is defined. Is there? Well, not much it would seem, since on the one hand we are considering the relationship between $\mathrm{Css}(\widehat{u})$ and $\mathrm{Css}(\widehat{v})$ and on the other the relationship between $\mathrm{Css}(u) \cap \mathbb{S}^{n-1}$ and $\mathrm{Css}(v) \cap \mathbb{S}^{n-1}$. If these are to be related, we would have to find a relationship of some sort between $\mathrm{Css}(u)$ and $\mathrm{Css}(\widehat{u})$. As we shall see, there is one but it is not very strong as can be guessed from Lemma 7.11. This is not so much a bad thing as a sign that we should look for another notion which combines aspects of both $\mathrm{Css}(u)$ and $\mathrm{Css}(\widehat{u})$. This we will do through the notion of *wave-front set*. In fact we define two related objects. The first is the more conventional, the second is more natural in our present discussion.

DEFINITION 7.12. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ we define the* wavefront set *of u to be*

$$\mathrm{WF}(u) = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{S}^{n-1};\\ \exists\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n),\ \phi(x) \neq 0,\ \omega \notin \mathrm{Css}(\widehat{\phi u})\}^{\complement} \tag{7.37}$$

and more generally the scattering wavefront set *by*

$$\mathrm{WF}_{\mathrm{sc}}(u) = \mathrm{WF}(u) \cup \{(\omega, p) \in \mathbb{S}^{n-1} \times \mathbb{B}^n;\\ \exists\ \psi \in \mathcal{C}^\infty(\mathbb{S}^n),\ \psi(\omega) \neq 0,\ R > 0 \text{ such that } p \notin \mathrm{Css}(\widehat{\psi_R u})\}^{\complement}. \tag{7.38}$$

So, the definition is really always the same. To show that $(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ we need to find 'a cutoff Φ near p' – depending on whether $p \in \mathbb{R}^n$ or $p \in \mathbb{S}^{n-1}$ this is either $\Phi = \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $F = \phi(p) \neq 0$ or a ψ_R where $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ has $\psi(p) \neq 0$ – such that $q \notin \mathrm{Css}(\widehat{\Phi u})$. One crucial property is

LEMMA 7.13. *If $(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ then if $p \in \mathbb{R}^n$ there exists a neighbourhood $U \subset \mathbb{R}^n$ of p and a neighbourhood $U \subset \mathbb{B}^n$ of q such that for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with support in U, $U' \cap \mathrm{Css}(\widehat{\phi u}) = \emptyset$; similarly if $p \in \mathbb{S}^{n-1}$ then there exists a neigbourhood $\tilde{U} \subset \mathbb{B}^n$ of p such that $U' \cap \mathrm{Css}(\widehat{\psi_R u}) = \emptyset$ if $\mathrm{Csp}(\omega_R) \subset \tilde{U}$.*

PROOF. First suppose $p \in \mathbb{R}^n$. From the definition of conic singular support, (7.37) means precisely that there exists $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $\psi(\omega) \neq 0$ and R such that

$$\psi_R(\widehat{\phi u}) \in \mathcal{S}(\mathbb{R}^n). \tag{7.39}$$

Since we know that $\widehat{\phi u} \in \mathcal{C}^\infty(\mathbb{R}^n)$, this is actually true for all $R > 0$ as soon as it is true for one value. Furthermore, if $\phi' \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ has $\mathrm{supp}(\phi') \subset \{\phi \neq 0\}$ then $\omega \notin \mathrm{Css}(\widehat{\phi' u})$ follows from $\omega \notin \mathrm{Css}(\widehat{\phi u})$. Indeed we can then write $\phi' = \mu\phi$ where $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so it suffices to show that if $v \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ has $\omega \notin \mathrm{Css}(\widehat{v})$ then $\omega \notin \mathrm{Css}(\widehat{\mu v})$ if $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Since $\widehat{\mu v} = (2\pi)^{-n} \upsilon * \widehat{u}$ where $\check{\upsilon} = \widehat{\mu} \in \mathcal{S}(\mathbb{R}^n)$, applying Lemma 7.8 we see that $\mathrm{Css}(\upsilon * \widehat{v}) \subset \mathrm{Css}(\widehat{v})$, so indeed $\omega \notin \mathrm{Css}(\widehat{\phi' u})$.

The case that $p \in \mathbb{S}^{n-1}$ is similar. Namely we have one cut-off ψ_R with $\psi(p) \neq 0$ and $q \notin \mathrm{Css}(\widehat{\omega_R u})$. We can take $U = \{\psi_{R+10} \neq 0\}$ since if

$\psi'_{R'}$ has conic support in U then $\psi'_{R'} = \psi'' R' \psi_R$ for some $\psi'' \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$. Thus

$$\widehat{\psi'_{R'} u} = v * \widehat{\psi_R u},\ \check{v} = \widehat{\omega''_{R''}}. \tag{7.40}$$

From Lemma 7.11 and Corollary7.9 we deduce that

$$\operatorname{Css}(\widehat{\psi'_{R'} u}) \subset \operatorname{Css}(\widehat{\omega_R u}) \tag{7.41}$$

and hence the result follows with U' a small neighourhood of q. $\square$

Proposition 7.14. *For any $u \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\begin{aligned} \mathrm{WF}_{\mathrm{sc}}(u) \subset \partial(\mathbb{B}^n \times \mathbb{B}^n) &= (\mathbb{B}^n \times \mathbb{S}^{n-1}) \cup (\mathbb{S}^{n-1} \times \mathbb{B}^n) \\ &= (\mathbb{R}^n \times \mathbb{S}^{n-1}) \cup (\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \cup (\mathbb{S}^{n-1} \times \mathbb{R}^n) \end{aligned} \tag{7.42}$$

and $\mathrm{WF}(u) \subset \mathbb{R}^n$ are closed sets and under projection onto the first variable

$$\pi_1(\mathrm{WF}(u)) = \operatorname{singsupp}(u) \subset \mathbb{R}^n,\ \pi_1(\mathrm{WF}_{\mathrm{sc}}(u)) = \operatorname{Css}(u) \subset \mathbb{B}^n. \tag{7.43}$$

Proof. To prove the first part of (7.43) we need to show that if $(\bar{x}, \omega) \notin \mathrm{WF}(u)$ for all $\omega \in \mathbb{S}^{n-1}$ with $\bar{x} \in \mathbb{R}^n$ fixed, then $\bar{x} \notin \operatorname{singsupp}(u)$. The definition (7.37) means that for each $\omega \in \mathbb{S}^{n-1}$ there exists $\phi_\omega \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi_\omega(\bar{x}) \neq 0$ such that $\omega \notin \operatorname{Css}(\widehat{\phi_\omega u})$. Since $\operatorname{Css}(\phi u)$ is closed and $\mathbb{S}^{n-1}$ is compact, a finite number of these cutoffs, $\phi_j \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, can be chosen so that $\phi_j(\bar{x}) \neq 0$ with the $\mathbb{S}^{n-1} \setminus \operatorname{Css}(\widehat{\phi_j u})$ covering $\mathbb{S}^{n-1}$. Now applying Lemma 7.13 above, we can find one $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, with support in $\bigcap_j \{\phi_j(x) \neq 0\}$ and $\phi(\bar{x}) \neq 0$, such that $\operatorname{Css}(\widehat{\phi u}) \subset \operatorname{Css}(\widehat{\phi_j u})$ for each j and hence $\phi u \in \mathcal{S}(\mathbb{R}^n)$ (since it is already smooth). Thus indeed it follows that $\bar{x} \notin \operatorname{singsupp}(u)$. The converse, that $\bar{x} \notin \operatorname{singsupp}(u)$ implies $(\bar{x}, \omega) \notin \mathrm{WF}(u)$ for all $\omega \in \mathbb{S}^{n-1}$ is immediate.

The argument to prove the second part of (7.43) is similar. Since, by definition, $\mathrm{WF}_{\mathrm{sc}}(u) \cap (\mathbb{R}^n \times \mathbb{B}^n) = \mathrm{WF}(u)$ and $\operatorname{Css}(u) \cap \mathbb{R}^n = \operatorname{singsupp}(u)$ we only need consider points in $\operatorname{Css}(u) \cap \mathbb{S}^{n-1}$. Now, we first check that if $\theta \notin \operatorname{Css}(u)$ then $\{\theta\} \times \mathbb{B}^n \cap \mathrm{WF}_{\mathrm{sc}}(u) = \emptyset$. By definition of $\operatorname{Css}(u)$ there is a cut-off ψ_R, where $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $\psi(\theta) \neq 0$, such that $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$. From (7.38) this implies that $(\theta, p) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ for all $p \in \mathbb{B}^n$.

Now, Lemma 7.13 allows us to apply the same argument as used above for WF. Namely we are given that $(\theta, p) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ for all $p \in \mathbb{B}^n$. Thus, for each p we may find ψ_R, depending on p, such that $\psi(\theta) \neq 0$ and $p \notin \operatorname{Css}(\widehat{\psi_R u})$. Since $\mathbb{B}^n$ is compact, we may choose a finite subset of these conic localizers, $\psi^{(j)}_{R_j}$ such that the intersection of the corresponding sets $\mathrm{Css}(\widehat{\psi^{(j)}_{R_j}u})$, is empty, i.e. their complements cover $\mathbb{B}^n$. Now, using Lemma 7.13 we may choose one ψ with support in the intersection of the sets $\{\psi^{(j)} \neq 0\}$ with $\psi(\theta) \neq 0$ and one R such that $\mathrm{Css}(\widehat{\psi_R u}) = \emptyset$, but this just means that $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$ and so $\theta \notin \mathrm{Css}(u)$ as desired.

The fact that these sets are closed (in the appropriate sets) follows directly from Lemma7.13. $\square$

Corollary 7.15. *For* $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$\mathrm{WF}_{\mathrm{sc}}(u) = \emptyset \Longleftrightarrow u \in \mathcal{S}(\mathbb{R}^n). \tag{7.44}$$

Let me return to the definition of $\mathrm{WF}_{\mathrm{sc}}(u)$ and rewrite it, using what we have learned so far, in terms of a decomposition of u.

Proposition 7.16. *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $(p,q) \in \partial(\mathbb{B}^n \times \mathbb{B}^n)$,

$$\begin{gathered}(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u) \Longleftrightarrow \\ u = u_1 + u_2,\ u_1,\ u_2 \in \mathcal{S}'(\mathbb{R}^n),\ p \notin \mathrm{Css}(u_1),\ q \notin \mathrm{Css}(\widehat{u_2}).\end{gathered} \tag{7.45}$$

Proof. For given $(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u)$, take $\Phi = \phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi \equiv 1$ near p, if $p \in \mathbb{R}^n$ or $\Phi = \psi_R$ with $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $\psi \equiv 1$ near p, if $p \in \mathbb{S}^{n-1}$. In either case $p \notin \mathrm{Css}(u_1)$ if $u_1 = (1-\Phi)u$ directly from the definition. So $u_2 = u - u_1 = \Phi u$. If the support of Φ is small enough it follows as in the discussion in the proof of Proposition 7.14 that

$$q \notin \mathrm{Css}(\widehat{u_2}). \tag{7.46}$$

Thus we have (7.45) in the forward direction.

For reverse implication it follows directly that $(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u_1)$ and that $(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u_2)$. $\square$

This restatement of the definition makes it clear that there a high degree of symmetry under the Fourier transform

Corollary 7.17. *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$(p,q) \in \mathrm{WF}_{\mathrm{sc}}(u)) \Longleftrightarrow (q,-p) \in \mathrm{WF}_{\mathrm{sc}}(\hat{u}). \tag{7.47}$$

Proof. I suppose a corollary should not need a proof, but still .... The statement (7.47) is equivalent to

$$(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u)) \Longrightarrow (q,-p) \notin \mathrm{WF}_{\mathrm{sc}}(\hat{u}) \tag{7.48}$$

since the reverse is the same by Fourier inversion. By (7.45) the condition on the left is equivalent to $u = u_1 + u_2$ with $p \notin \mathrm{Css}(u_1)$, $q \notin \mathrm{Css}(\widehat{u_2})$. Hence equivalent to

$$\widehat{u} = v_1 + v_2,\ v_1 = \widehat{u_2},\ \widehat{v_2} = (2\pi)^{-n}\check{u}_1 \tag{7.49}$$

so $q \notin \mathrm{Css}(v_1)$, $-p \notin \mathrm{Css}(\widehat{v_2})$ which proves (7.47). $\square$

Now, we can exploit these notions to refine our conditions under which pairing, the product and convolution can be defined.

THEOREM 7.18. *For u, $v \in \mathcal{S}'(\mathbb{R}^n)$*

$$uv \in \mathcal{S}'(\mathbb{R}^n) \text{ is unambiguously defined provided} \\ (p,\omega) \in \mathrm{WF}_{\mathrm{sc}}(u) \cap (\mathbb{B}^n \times \mathbb{S}^{n-1}) \Longrightarrow (p,-\omega) \notin \mathrm{WF}_{\mathrm{sc}}(v) \tag{7.50}$$

and

$$u * v \in \mathcal{S}'(\mathbb{R}^n) \text{ is unambiguously defined provided} \\ (\theta,q) \in \mathrm{WF}_{\mathrm{sc}}(u) \cap (\mathbb{S}^{n-1} \times \mathbb{B}^n) \Longrightarrow (-\theta,q) \notin \mathrm{WF}_{\mathrm{sc}}(v). \tag{7.51}$$

PROOF. Let us consider convolution first. The hypothesis, (7.51) means that for each $\theta \in \mathbb{S}^{n-1}$

$$\{q \in \mathbb{B}^{n-1}; (\theta,q) \in \mathrm{WF}_{\mathrm{sc}}(u)\} \cap \{q \in \mathbb{B}^{n-1}; (-\theta,q) \in \mathrm{WF}_{\mathrm{sc}}(v)\} = \emptyset. \tag{7.52}$$

Now, the fact that $\mathrm{WF}_{\mathrm{sc}}$ is always a closed set means that (7.52) remains true near θ in the sense that if $U \subset \mathbb{S}^{n-1}$ is a sufficiently small neighbourhood of θ then

$$\{q \in \mathbb{B}^{n-1}; \exists\, \theta' \in U,\ (\theta',q) \in \mathrm{WF}_{\mathrm{sc}}(u)\} \\ \cap \{q \in \mathbb{B}^{n-1}; \exists\, \theta'' \in U,\ (-\theta'',q) \in \mathrm{WF}_{\mathrm{sc}}(v)\} = \emptyset. \tag{7.53}$$

The compactness of $\mathbb{S}^{n-1}$ means that there is a finite cover of $\mathbb{S}^{n-1}$ by such sets U_j. Now select a partition of unity ψ_i of $\mathbb{S}^{n-1}$ which is not only subordinate to this open cover, so each ψ_i is supported in one of the U_j but satisfies the additional condition that

$$\mathrm{supp}(\psi_i) \cap (-\mathrm{supp}(\psi_{i'})) \neq \emptyset \Longrightarrow \\ \mathrm{supp}(\psi_i) \cup (-\mathrm{supp}(\psi_{i'})) \subset U_j \text{ for some } j. \tag{7.54}$$

Now, if we set $u_i = (\psi_i)_R u$, and $v_{i'} = (\psi_{i'})_R v$, we know that $u - \sum\limits_i u_i$ has compact support and similarly for v. Since convolution is already known to be possible if (at least) one factor has compact support, it suffices to define $u_i * v_{i'}$ for every i, i'. So, first suppose that $\mathrm{supp}(\psi_i) \cap (-\mathrm{supp}(\psi_{i'})) \neq \emptyset$. In this case we conclude from (7.54) that

$$\mathrm{Css}(\widehat{u_i}) \cap \mathrm{Css}(\widehat{v_{i'}}) = \emptyset. \tag{7.55}$$

Thus we may *define*

$$\widehat{u_i * v_{i'}} = \widehat{u}_i \widehat{v}_{i'} \tag{7.56}$$

using (7.20). On the other hand if $\operatorname{supp}\psi_i \cap (-\operatorname{supp}(\psi_{i'})) = \emptyset$ then

$$\operatorname{Css}(u_i) \cap (-\operatorname{Css}(v_{i'})) \cap \mathbb{S}^{n-1} = \emptyset \tag{7.57}$$

and in this case we can define $u_i * v_{i'}$ using Lemma 7.10.

Thus with such a decomposition of u and v all terms in the convolution are well-defined. Of course we should check that this definition is independent of choices made in the decomposition. I leave this to you.

That the product is well-defined under condition (7.50) now follows if we define it using convolution, i.e. as

$$\widehat{uv} = f * g,\ \ f = \widehat{u},\ \ \check{g} = \widehat{v}. \tag{7.58}$$

Indeed, using (7.47), (7.50) for u and v becomes (7.51) for f and g. $\square$

8. Homogeneous distributions

Next time I will talk about homogeneous distributions. On $\mathbb{R}$ the functions

$$x_t^s = \begin{cases} x^s & x > 0 \\ 0 & x < 0 \end{cases}$$

where $S \in \mathbb{R}$, is locally integrable (and hence a tempered distribution) precisely when $S > -1$. As a function it is homogeneous of degree s. Thus if $a > 0$ then

$$(ax)_t^s = a^s x_t^s\,.$$

Thinking of $x_t^s = \mu_s$ as a distribution we can set this as

$$\begin{aligned} \mu_s(ax)(\varphi) &= \int \mu_s(ax)\varphi(x)\,dx \\ &= \int \mu_s(x)\varphi(x/a)\frac{dx}{a} \\ &= a^s\mu_s(\varphi)\,. \end{aligned}$$

Thus if we *define* $\varphi_a(x) = \frac{1}{a}\varphi(\frac{x}{a})$, for any $a > 0$, $\varphi \in \mathcal{S}(\mathbb{R})$ we can ask whether a distribution is homogeneous:

$$\mu(\varphi_a) = a^s\mu(\varphi)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}).$$

9. Operators and kernels

From here on a summary of parts of 18.155 used in 18.156 – to be redistributed backwards With some corrections by incorporated.

10. Fourier transform

The basic properties of the Fourier transform, tempered distributions and Sobolev spaces form the subject of the first half of this course. I will recall and slightly expand on such a standard treatment.

11. Schwartz space.

The space $\mathcal{S}(\mathbb{R}^n)$ of all complex-volumed functions with rapidly decreasing derivatives of all orders is a complete metric space with metric

$$
\begin{gathered}
d(u,v) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|u-v\|_{(k)}}{1+\|u-v\|_{(k)}} \text{ where} \\
\|u\|_{(k)} = \sum_{|\alpha|+|\beta|\le k} \sup_{z\in\mathbb{R}^n} |z^\alpha D_z^\beta u(z)|.
\end{gathered}
\tag{11.1}
$$

Here and below I will use the notation for derivatives

$$D_z^\alpha = D_{z_1}^{\alpha_1} \dots, D_{z_n}^{\alpha_n},\ D_{z_j} = \frac{1}{i} 1 \frac{\partial}{\partial z_j}.$$

These norms can be replaced by other equivalent ones, for instance by reordering the factors

$$\|u\|'_{(k)} = \sum_{|\alpha|+|\beta|\le k} \sup_{z\in\mathbb{R}^n} |D_z^\beta (z^\beta u)|.$$

In fact it is only the cumulative effect of the norms that matters, so one can use

$$\|u\|''_{(k)} = \sup_{z\in\mathbb{R}^n} |\langle z\rangle^{2k} (\Delta+1)^k u| \tag{11.2}$$

in (11.1) and the same topology results. Here

$$\langle z\rangle^2 = 1+|z|^2,\ \Delta = \sum_{j=1}^n D_j^2$$

(so the Laplacian is formally positive, the geometers' convention). It is not quite so trivial to see that inserting (11.2) in (11.1) gives an equivalent metric.

12. Tempered distributions.

The space of (metrically) continuous linear maps

$$f : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{12.1}$$

is the space of tempered distribution, denoted $\mathcal{S}'(\mathbb{R}^n)$ since it is the dual of $\mathcal{S}(\mathbb{R}^n)$. The continuity in (12.1) is equivalent to the estimates

$$\exists\ k,\ C_k > 0 \text{ s.t. } |f(\varphi)| \le C_k \|\varphi\|_{(k)}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{12.2}$$

There are several topologies which can be considered on $\mathcal{S}'(\mathbb{R}^n)$. Unless otherwise noted we consider the *uniform topology* on $\mathcal{S}'(\mathbb{R}^n)$; a subset $U \subset \mathcal{S}'(\mathbb{R}^n)$ is open in the uniform topology if for every $u \in U$ and every k sufficiently large there exists $\delta_k > 0$ (both k and δ_k depending on u) such that

$$v \in \mathcal{S}'(\mathbb{R}^n),\ |(u-u)(\varphi) \le \delta_k \|\varphi\|_{(k)} \Rightarrow v \in U.$$

For linear maps it is straightforward to work out continuity conditions. Namely

$$\begin{gathered}
P : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^m)\\
Q : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m)\\
R : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^m)\\
S : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m)
\end{gathered}$$

are, respectively, continuous for the metric and uniform topologies if

$\forall\ k\ \exists\ k',\ C \text{ s.t. } \|P\varphi\|_{(k)} \le C\|\varphi\|_{(k')}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)$

$\exists\ k,\ k',\ C \text{ s.t. } |Q\varphi(\psi)| \le C\|\varphi\|_{(k)}\|\psi\|_{(k')}$

$\forall\ k,\ k'\ \exists\ C \text{ s.t. } |u(\varphi)| \le \|\varphi\|_{(k')}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \|Ru\|_{(k)} \le C$

$\forall\ k'\ \exists\ k,\ C,\ C' \text{ s.t. } \|u(\varphi)\|_{(k)} \le \|\varphi\|_{(k)} \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow |Su(\psi)| \le C'\|\psi\|_{(k')}\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n).$

The particular case of R, for $m = 0$, where at least formally $\mathcal{S}(\mathbb{R}^0) = \mathbb{C}$, corresponds to the reflexivity of $\mathcal{S}(\mathbb{R}^n)$, that

$$\begin{gathered}
R : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathbb{C} \text{ is cts. iff } \exists\ \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ s.t.}\\
Ru = u(\varphi) \text{ i.e. } (\mathcal{S}'(\mathbb{R}^n))' = \mathcal{S}(\mathbb{R}^n).
\end{gathered}$$

In fact another extension of the middle two of these results corresponds to the Schwartz kernel theorem:

$Q : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m)$ is linear and continuous

  iff $\exists\ Q \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n) \text{ s.t. } (Q(\varphi))(\psi) = Q(\psi \boxtimes \varphi)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^m)\ \psi \in \mathcal{S}(\mathbb{R}^n).$

$R : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is linear and continuous

  iff $\exists\ R \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n) \text{ s.t. } (Ru)(z) = u(R(z,\cdot)).$

Schwartz test functions are dense in tempered distributions

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

where the *standard inclusion* is via Lebesgue measure

$$\mathcal{S}(\mathbb{R}^n) \ni \varphi \mapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n),\ u_\varphi(\psi) = \int_{\mathbb{R}^n} \varphi(z)\psi(z)dz. \tag{12.3}$$

The basic operators of differentiation and multiplication are transferred to $\mathcal{S}'(\mathbb{R}^n)$ by duality so that they remain consistent with the (12.3):

$$\begin{aligned} D_z u(\varphi) &= u(-D_z\varphi) \\ fu(\varphi) &= u(f\varphi)\ \forall\ f \in \mathcal{S}(\mathbb{R}^n)). \end{aligned}$$

In fact multiplication extends to the space of function of polynomial growth:

$$\forall\ \alpha \in \mathbb{N}_0^n\ \exists\ k \text{ s.t. } |D_z^\alpha f(z)| \le C\langle z\rangle^k.$$

Thus such a function is a multiplier on $\mathcal{S}(\mathbb{R}^n)$ and hence by duality on $\mathcal{S}'(\mathbb{R}^n)$ as well.

13. Fourier transform

Many of the results just listed are best proved using the Fourier transform

$$\begin{gathered} \mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \\ \mathcal{F}\varphi(\zeta) = \hat{\varphi}(\zeta) = \int e^{-iz\zeta}\varphi(z)dz. \end{gathered}$$

This map is an isomorphism that extends to an isomorphism of $\mathcal{S}'(\mathbb{R}^n)$

$$\begin{gathered} \mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \\ \mathcal{F}\varphi(D_{z_j}u) = \zeta_j\mathcal{F}u,\ \mathcal{F}(z_ju) = -D_{\zeta_j}\mathcal{F}u \end{gathered}$$

and also extends to an isomorphism of $L^2(\mathbb{R}^n)$ from the dense subset

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^2)\mathrm{d}ense,\ \|\mathcal{F}\varphi\|^2_{L^2} = (2\pi)^n\|\varphi\|^2_{L^2}. \tag{13.1}$$

14. Sobolev spaces

Plancherel's theorem, (**??**), is the basis of the definition of the (standard, later there will be others) Sobolev spaces.

$$\begin{gathered} H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n);\ (1+|\zeta|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)\} \\ \|u\|_s^2 = \int_{\mathbb{R}^n} (1+|\zeta|^2)^s|\hat{u}(\zeta)|d\zeta, \end{gathered}$$

where we use the fact that $L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is a well-defined injection (regarded as an inclusion) by continuous extension from (12.3). Now,

$$D^\alpha : H^s(\mathbb{R}^n) \longrightarrow H^{s-|\alpha|}(\mathbb{R}^n)\ \forall\ s,\ \alpha. \tag{14.1}$$

As well as this action by constant coefficient differential operators we note here that multiplication by Schwartz functions also preserves the Sobolev spaces – this is generalized with a different proof below. I give this cruder version first partly to show a little how to estimate convolution integrals.

PROPOSITION 14.1. *For any $s \in \mathbb{R}$ there is a continuous bilinear map extending multiplication on Schwartz space*

$$\mathcal{S}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n) \tag{14.2}$$

PROOF. The product ϕu is well-defined for any $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Since Schwartz functions are dense in the Sobolev spaces it suffices to assume $u \in \mathcal{S}(\mathbb{R}^n)$ and then to use continuity. The Fourier transform of the product is the convolution of the Fourier transforms

$$\widehat{\phi u} = (2\pi)^{-n}\hat{\phi} * \hat{u},\ \hat{\phi} * \hat{u}(\xi) = \int_{\mathbb{R}^n} \hat{\phi}(\xi - \eta)\hat{u}(\eta)d\eta. \tag{14.3}$$

This is proved above, but let's just note that in this case it is easy enough since all the integrals are absolutely convergent and we can compute the inverse Fourier transform of the convolution

$$\begin{aligned}
&(2\pi)^{-n} \int d\xi e^{iz\cdot\xi} \int_{\mathbb{R}^n} \hat{\phi}(\xi - \eta)\hat{u}(\eta)d\eta \\
&\qquad = (2\pi)^{-n} \int d\xi e^{iz\cdot(\xi-\eta)} \int_{\mathbb{R}^n} \hat{\phi}(\xi - \eta)e^{iz\cdot\eta}\hat{u}(\eta)d\eta \\
&\qquad = (2\pi)^{-n} \int d\Xi e^{iz\cdot\Xi} \int_{\mathbb{R}^n} \hat{\phi}(\Xi)e^{iz\cdot\eta}\hat{u}(\eta)d\eta \\
&\qquad = (2\pi)^n\phi(z)u(z).
\end{aligned} \tag{14.4}$$

First, take $s = 0$ and prove this way the, rather obvious, fact that $\mathcal{S}$ is a space of multipliers on L^2. Writing out the square of the absolute value of the integral as the product with the complex conjugate, estimating by the absolute value and then using the Cauchy-Schwarz inequality gives what we want

$$\begin{aligned}
&|\int |\int \psi(\xi - \eta)\hat{u}(\eta)d\eta|^2 d\xi \\
&\qquad \leq \int\int |\psi(\xi - \eta_1)||\hat{u}(\eta_1)||\psi(\xi - \eta_2)||\hat{u}(\eta_2)|d\eta_1 d\eta_2 d\xi \\
&\qquad \leq \int\int |\psi(\xi - \eta_1)||\psi(\xi - \eta_2)||\hat{u}(\eta_2)|^2 d\eta_1 d\eta_2 \\
&\qquad \leq (\int |\psi|)^2 \|u\|_{L^2}^2.
\end{aligned} \tag{14.5}$$

Here, we have decomposed the integral as the product of $|\psi(\xi-\eta_1)|^{\frac{1}{2}}|\hat{u}(\eta_1)||\psi(\xi-\eta_2)|^{\frac{1}{2}}$ and the same term with the η variables exchanged. The two resulting factors are then the same after changing variable so there is no square-root in the integral.

Note that what we have really shown here is the well-known result:-

Lemma 14.2. *Convolution gives is a continous bilinear map*

$$L^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \ni (u, v) \longmapsto u * v \in L^2(\mathbb{R}^n), \ \|u * v\|_{L^2} \le \|u\|_{L^1}\|v\|_{L^2}. \tag{14.6}$$

Now, to do the general case we need to take care of the weights in the integral for the Sobolev norm

$$\|\phi u\|^2_{H^s} = \int (1 + |\xi|^2)^s |\widehat{\phi u}(\xi)|^2 d\xi. \tag{14.7}$$

To do so, we divide the convolution integral into two regions:-

$$\begin{aligned} I &= \{\eta \in \mathbb{R}^n; |\xi - \eta| \ge \frac{1}{10}(|\xi| + |\eta|)\} \\ II &= \{\eta \in \mathbb{R}^n; |\xi - \eta| \le \frac{1}{10}(|\xi| + |\eta|)\}. \end{aligned} \tag{14.8}$$

In the first region $\phi(\xi - \eta)$ is rapidly decreasing in both variable, so

$$|\psi(\xi - \eta)| \le C_N(1 + |\xi|)^{-N}(1 + |\eta|)^{-N} \tag{14.9}$$

for any N and as a result this contribution to the integral is rapidly decreasing:-

$$|\int_I \psi(\xi - \eta)\hat{u}(\eta)d\eta| \le C_N(1 + |\xi|)^{-n}\|u\|_{H^s} \tag{14.10}$$

where the η decay is used to squelch the weight. So this certainly constributes a term to $\psi * \hat{u}$ with the bilinear bound.

To estimate the contribution from the second region, proceed as above but the insert the weight after using the Cauchy-Schwartz intequality

$$\begin{aligned} &\int (1 + |\xi|^2)^s |\int_{II} \psi(\xi - \eta)\hat{u}(\eta)d\eta|^2 d\xi \\ &\le \int (1 + |\xi|^2)^s \int_{II}\int_{II} |\psi(\xi - \eta_1)||\hat{u}(\eta_1)||\psi(\xi - \eta_2)||\hat{u}(\eta_2)| d\eta_1 d\eta_2 \\ &\le \int\int_{II}\int_{II} (1 + |\xi|^2)^s (1 + |\eta_2|^2)^{-s} |\psi(\xi - \eta_1)||\psi(\xi - \eta_2)|(1 + |\eta_2|^2)^s |\hat{u}(\eta_2)|^2 d\eta_1 d\eta_2 \end{aligned} \tag{14.11}$$

Exchange the order of integration and note that in region II the two variables η_2 and ξ are each bounded relative to the other. Thus the quotient of the weights is bounded above so the same argument applies to estimate the integral by

$$C\left(\int d\Xi|\psi(\Xi)|\right)^2 \|u\|_{H^s}^2 \tag{14.12}$$

as desired. □

The Sobolev spaces are Hilbert spaces, so their duals are (conjugate) isomorphic to themselves. However, in view of our inclusion $L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, we habitually identify

$$(H^s(\mathbb{R}^n))' = H^{-s}(\mathbb{R}^n),$$

with the 'extension of the L^2 paring'

$$(u,v) = \text{``}\int u(z)v(z)dz\text{''} = (2\pi)^{-n}\int_{\mathbb{R}^n}\langle\zeta\rangle^s\hat{u}\cdot\langle\zeta\rangle^{-s}\hat{u}d\zeta.$$

Note that then (14) is a linear, not a conjugate-linear, isomorphism since (14) is a real pairing.

The Sobolev spaces decrease with increasing s,

$$H^s(\mathbb{R}^n) \subset H^{s'}(\mathbb{R}^n)\ \forall\ s \geq s'.$$

One essential property is the relationship between the 'L^2 derivatives' involved in the definition of Sobolev spaces and standard derivatives. Namely, the Sobolev embedding theorem:

$$\begin{aligned} s > \frac{n}{2} \Longrightarrow & H^s(\mathbb{R}^n) \subset \mathcal{C}^0_\infty(\mathbb{R}^n) \\ & = \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ its continuous and bounded}\}. \\ s > \frac{n}{2} + k,\ k \in \mathbb{N} \Longrightarrow & H^s(\mathbb{R}^n) \subset \mathcal{C}^k_\infty(\mathbb{R}^n) \\ & \overset{\text{def}}{=} \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ s.t. } D^\alpha u \in \mathcal{C}^0_\infty(\mathbb{R}^n)\ \forall\ |\alpha| \leq k\}. \end{aligned}$$

For positive integral s the Sobolev norms are easily written in terms of the functions, without Fourier transform:

$$\begin{gathered} u \in H^k(\mathbb{R}^n) \Leftrightarrow D^\alpha u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| \leq k \\ \|u\|_k^2 = \sum_{|\alpha|\leq k}\int_{\mathbb{R}^n}|D^\alpha u|^2 dz. \end{gathered}$$

For negative integral orders there is a similar characterization by duality, namely

$$\begin{aligned} H^{-k}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \text{ s.t. }, \exists\ u_\alpha \in L^2(\mathbb{R}^n),\ |\alpha| \geq k \\ u = \sum_{|\alpha|\leq k} D^\alpha u_\alpha\}. \end{aligned}$$

In fact there are similar "Hölder" characterizations in general. For $0 < s < 1$, $u \in H^s(\mathbb{R}^n) \Longrightarrow u \in L^2(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^{2n}} \frac{|u(z) - u(z')|^2}{|z - z'|^{n+2s}} dz dz' < \infty. \tag{14.13}$$

Then for $k < s < k+1$, $k \in \mathbb{N}$ $u \in H^s(\mathbb{R}^2)$ is equivalent to $D^\alpha \in H^{s-k}(\mathbb{R}^n)$ for all $|\alpha| \in k$, with corresponding (Hilbert) norm. Similar realizations of the norms exist for $s < 0$.

One simple consequence of this is that

$$\mathcal{C}^\infty_\infty(\mathbb{R}^n) = \bigcap_k \mathcal{C}^k_\infty(\mathbb{R}^n) = \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ s.t. } |D^\alpha u| \text{ is bounded } \forall\ \alpha\}$$

is a multiplier on *all* Sobolev spaces

$$\mathcal{C}^\infty_\infty(\mathbb{R}^n) \cdot H^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)\ \forall\ s \in \mathbb{R}.$$

15. Weighted Sobolev spaces.

It follows from the Sobolev embedding theorem that

$$\bigcap_s H^s(\mathbb{R}^n) \subset \mathcal{C}^\infty_\infty(\mathbb{R}^n); \tag{15.1}$$

in fact the intersection here is quite a lot smaller, but nowhere near as small as $\mathcal{S}(\mathbb{R}^n)$. To discuss decay at infinity, as will definitely want to do, we may use weighted Sobolev spaces.

The ordinary Sobolev spaces do not effectively define decay (or growth) at infinity. We will therefore also set

$$\begin{aligned} H^{m,l}(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n);\ \langle z \rangle^\ell u \in H^m(\mathbb{R}^n)\},\ m, \ell \in \mathbb{R}, \\ &= \langle z \rangle^{-\ell} H^m(\mathbb{R}^n)\,, \end{aligned}$$

where the second notation is supported to indicate that $u \in H^{m,l}(\mathbb{R}^n)$ may be written as a product $\langle z \rangle^{-\ell} v$ with $v \in H^m(\mathbb{R}^n)$. Thus

$$H^{m,\ell}(\mathbb{R}^n) \subset H^{m',\ell'}(\mathbb{R}^n) \text{ if } m \geq m' \text{ and } \ell \geq \ell',$$

so the spaces are decreasing in each index. As consequences of the *Schwartz structure theorem*

$$\begin{aligned} \mathcal{S}'(\mathbb{R}^n) &= \bigcup_{m,\ell} H^{m,\ell}(\mathbb{R}^n) \\ \mathcal{S}(\mathbb{R}^n) &= \bigcap_{m,\ell} H^{m,\ell}(\mathbb{R}^n). \end{aligned} \tag{15.2}$$

This is also true 'topologically' meaning that the first is an 'inductive limit' and the second a 'projective limit'.

Similarly, using some commutation arguments

$$D_{z_j} : H^{m,\ell}(\mathbb{R}^n) \longrightarrow H^{m-1,\ell}(\mathbb{R}^n),\ \forall\ m,\ elll$$
$$\times z_j : H^{m,\ell}(\mathbb{R}^n) \longrightarrow H^{m,\ell-1}(\mathbb{R}^n).$$

Moreover there is symmetry under the Fourier transform

$$\mathcal{F} : H^{m,\ell}(\mathbb{R}^n) \longrightarrow H^{\ell,m}(\mathbb{R}^n) \text{ is an isomorphism } \forall\ m,\ \ell.$$

As with the usual Sobolev spaces, $\mathcal{S}(\mathbb{R}^n)$ is dense in all the $H^{m,\ell}(\mathbb{R}^n)$ spaces and the continuous extension of the L^2 paring gives an identification

$$\begin{aligned} H^{m,\ell}(\mathbb{R}^n) \cong (H^{-m,-\ell}(\mathbb{R}^n))' \text{ fron} \\ H^{m,\ell}(\mathbb{R}^n) \times H^{-m,-\ell}(\mathbb{R}^n) \ni u, v \mapsto \\ (u, v) = \text{“} \int u(z)v(z)dz''. \end{aligned}$$

Let R_s be the operator defined by Fourier multiplication by $\langle\zeta\rangle^s$:

$$R_s : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\ \widehat{R_s f}(\zeta) = \langle\zeta\rangle^s \hat{f}(\zeta). \tag{15.3}$$

Lemma 15.1. *If $\psi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$M_s = [\psi, R_s *] : H^t(\mathbb{R}^n) \longrightarrow H^{t-s+1}(\mathbb{R}^n) \tag{15.4}$$

is bounded for each t.

Proof. Since the Sobolev spaces are defined in terms of the Fourier transform, first conjugate and observe that (15.4) is equivalent to the boundeness of the integral operator with kernel

$$K_{s,t}(\zeta, \zeta') = (1+|\zeta|^2)^{\frac{t-s+1}{2}} \hat{\psi}(\zeta-\zeta') \left((1 + |\zeta'|^2)^{\frac{s}{2}} - (1 + |\zeta|^2)^{\frac{s}{2}}\right) (1+|\zeta'|^2)^{-\frac{t}{2}} \tag{15.5}$$

on $L^2(\mathbb{R}^n)$. If we insert the characteristic function for the region near the diagonal

$$|\zeta - \zeta'| \le \frac{1}{4}(|\zeta| + |\zeta'|) \Longrightarrow |\zeta| \le 2|\zeta'|,\ |\zeta'| \le 2|\zeta| \tag{15.6}$$

then $|\zeta|$ and $|\zeta'|$ are of comparable size. Using Taylor's formula

$$\begin{aligned} (1+|\zeta'|^2)^{\frac{s}{2}} - (1+|\zeta|^2)^{\frac{s}{2}} = s(\zeta-\zeta')\cdot \int_0^1 (t\zeta+(1-t\zeta')\left(1 + |t\zeta + (1-t)\zeta'|^2\right)^{\frac{s}{2}-1} dt \\ \Longrightarrow \left|(1 + |\zeta'|^2)^{\frac{s}{2}} - (1 + |\zeta|^2)^{\frac{s}{2}}\right| \le C_s|\zeta - \zeta'|(1 + |\zeta|)^{s-1}. \end{aligned} \tag{15.7}$$

It follows that in the region (15.6) the kernel in (15.5) is bounded by

$$C|\zeta - \zeta'||\hat{\psi}(\zeta - \zeta')|. \tag{15.8}$$

In the complement to (15.6) the kernel is rapidly decreasing in ζ and ζ' in view of the rapid decrease of $\hat{\psi}$. Both terms give bounded operators on L^2, in the first case using the same estimates that show convolution by an element of $\mathcal{S}$ to be bounded. $\square$

LEMMA 15.2. *If $u \in H^s(\mathbb{R}^n)$ and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then*

$$\|\psi u\|_s \le \|\psi\|_{L^\infty}\|u\|_s + C\|u\|_{s-1} \tag{15.9}$$

where the constant depends on s and ψ but not u.

PROOF. This is really a standard estimate for Sobolev spaces. Recall that the Sobolev norm is related to the L^2 norm by

$$\|u\|_s = \|\langle D\rangle^s u\|_{L^2}. \tag{15.10}$$

Here $\langle D\rangle^s$ is the convolution operator with kernel defined by its Fourier transform

$$\langle D\rangle^s u = R_s * u,\ \widehat{R_s}(\zeta) = (1+|\zeta|^2)^{\frac{s}{2}}. \tag{15.11}$$

To get (15.9) use Lemma 15.1.

From (15.4), (writing 0 for the L^2 norm)

$$\begin{aligned}\|\psi u\|_s &= \|R_s * (\psi u)\|_0 \le \|\psi(R_s * u)\|_0 + \|M_s u\|_0 \\ &\le \|\psi\|_{L^\infty}\|R_s u\|_0 + C\|u\|_{s-1} \le \|\psi\|_{L^\infty}\|u\|_s + C\|u\|_{s-1}.\end{aligned} \tag{15.12}$$

This completes the proof of (15.9) and so of Lemma 15.2. $\square$

16. Multiplicativity

Of primary importance later in our treatment of non-linear problems is some version of the multliplicative property

$$A^s(\mathbb{R}^n) = \begin{cases} H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) & s \le \frac{n}{2} \\ H^s(\mathbb{R}^n) & s > \frac{n}{2} \end{cases} \text{ is a } \mathcal{C}^\infty \text{ algebra.} \tag{16.1}$$

Here, a $\mathcal{C}^\infty$ algebra is an algebra with an additional closure property. Namely if $F : \mathbb{R}^N \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ function vanishing at the origin and $u_1, \dots, u_N \in A^s$ are *real-valued* then

$$F(u_1, \dots, u_n) \in A^s.$$

I will only consider the case of real interest here, where s is an integer and $s > \frac{n}{2}$. The obvious place to start is

LEMMA 16.1. *If $s > \frac{n}{2}$ then*

$$u,\ v \in H^s(\mathbb{R}^n) \Longrightarrow uv \in H^s(\mathbb{R}^n). \tag{16.2}$$

Proof. We will prove this directly in terms of convolution. Thus, in terms of weighted Sobolev spaces $u \in H^s(\mathbb{R}^n) = H^{s,0}(\mathbb{R}^n)$ is equivalent to $\hat{u} \in H^{0,s}(\mathbb{R}^n)$. So (16.2) is equivalent to

$$u,\ v \in H^{0,s}(\mathbb{R}^n) \Longrightarrow u * v \in H^{0,s}(\mathbb{R}^n). \tag{16.3}$$

Using the density of $\mathcal{S}(\mathbb{R}^n)$ it suffices to prove the estimate

$$\|u * v\|_{H^{0,s}} \le C_s \|u\|_{H^{0,s}} \|v\|_{H^{0,s}} \text{ for } s > \frac{n}{2}. \tag{16.4}$$

Now, we can write $u(\zeta) = \langle\zeta\rangle^{-s} u'$ etc and convert (16.4) to an estimate on the L^2 norm of

$$\langle\zeta\rangle^{-s} \int \langle\xi\rangle^{-s} u'(\xi) \langle\zeta - \xi\rangle^{-s} v'(\zeta - \xi) d\xi \tag{16.5}$$

in terms of the L^2 norms of u' and $v' \in \mathcal{S}(\mathbb{R}^n)$.

Writing out the L^2 norm as in the proof of Lemma 15.1 above, we need to estimate the absolute value of

(16.6)

$$\int\int\int d\zeta d\xi d\eta \langle\zeta\rangle^{2s} \langle\xi\rangle^{-s} u_1(\xi) \langle\zeta - \xi\rangle^{-s} v_1(\zeta - \xi) \langle\eta\rangle^{-s} u_2(\eta) \langle\zeta - \eta\rangle^{-s} v_2(\zeta - \eta)$$

in terms of the L^2 norms of the u_i and v_i. To do so divide the integral into the four regions,

$$\begin{aligned}
&|\zeta - \xi| \le \frac{1}{4}(|\zeta| + |\xi|),\ |\zeta - \eta| \le \frac{1}{4}(|\zeta| + |\eta|)\\
&|\zeta - \xi| \le \frac{1}{4}(|\zeta| + |\xi|),\ |\zeta - \eta| \ge \frac{1}{4}(|\zeta| + |\eta|)\\
&|\zeta - \xi| \ge \frac{1}{4}(|\zeta| + |\xi|),\ |\zeta - \eta| \le \frac{1}{4}(|\zeta| + |\eta|)\\
&|\zeta - \xi| \ge \frac{1}{4}(|\zeta| + |\xi|),\ |\zeta - \eta| \ge \frac{1}{4}(|\zeta| + |\eta|).
\end{aligned} \tag{16.7}$$

Using (15.6) the integrand in (16.6) may be correspondingly bounded by

$$\begin{gathered}
C\langle\zeta - \eta\rangle^{-s} |u_1(\xi)||v_1(\zeta - \xi)| \cdot \langle\zeta - \xi\rangle^{-s} |u_2(\eta)||v_2(\zeta - \eta)|\\
C\langle\eta\rangle^{-s} |u_1(\xi)||v_1(\zeta - \xi)| \cdot \langle\zeta - \xi\rangle^{-s} |u_2(\eta)||v_2(\zeta - \eta)|\\
C\langle\zeta - \eta\rangle^{-s} |u_1(\xi)||v_1(\zeta - \xi)| \cdot \langle\xi\rangle^{-s} |u_2(\eta)||v_2(\zeta - \eta)|\\
C\langle\eta\rangle^{-s} |u_1(\xi)|v_1(\zeta - \xi)| \cdot \langle\xi\rangle^{-s} |u_2(\eta)||v_2(\zeta - \eta)|.
\end{gathered} \tag{16.8}$$

Now applying Cauchy-Schwarz inequality, with the factors as indicated, and changing variables appropriately gives the desired estimate. $\square$

Next, we extend this argument to (many) more factors to get the following result which is close to the Gagliardo-Nirenberg estimates (since I am concentrating here on L^2 methods I will not actually discuss the latter).

LEMMA 16.2. *If $s > \frac{n}{2}$, $N \geq 1$ and $\alpha_i \in \mathbb{N}_0^k$ for $i = 1, \dots, N$ are such that*

$$\sum_{i=1}^{N} |\alpha_i| = T \leq s$$

then

(16.9)

$$u_i \in H^s(\mathbb{R}^n) \Longrightarrow U = \prod_{i=1}^{N} D^{\alpha_i} u_i \in H^{s-T}(\mathbb{R}^n), \ \|U\|_{H^{s-T}} \leq C_N \prod_{i=1}^{N} \|u_i\|_{H^s}.$$

PROOF. We proceed as in the proof of Lemma 16.1 using the Fourier transform to replace the product by the convolution. Thus it suffices to show that

$$u_1 * u_2 * u_3 * \cdots * u_N \in H^{0,s-T} \text{ if } u_i \in H^{0,s-\alpha_i}. \tag{16.10}$$

Writing out the convolution symmetrically in all variables,

$$u_1 * u_2 * u_3 * \cdots * u_N(\zeta) = \int_{\zeta = \sum\limits_i \xi_i} u_1(\xi_1) \cdots u_N(\xi_N) \tag{16.11}$$

it follows that we need to estimate the L^2 norm in ζ of

$$\langle \zeta \rangle^{s-T} \int_{\zeta = \sum\limits_i \xi_i} \langle \xi_1 \rangle^{-s+a_1} v_1(\xi_1) \cdots \langle \xi_N \rangle^{-s+a_N} v_N(\xi_N) \tag{16.12}$$

for N factors v_i which are in L^2 with the $a_i = |\alpha|_i$ non-negative integers summing to $T \leq s$. Again writing the square as the product with the complex conjuage it is enough to estimate integrals of the type

$$\begin{gathered}\int_{\{(\xi,\eta) \in \mathbb{R}^{2N}; \sum\limits_i \xi_i = \sum\limits_i \eta_i\}} \langle \sum_i \xi \rangle^{2s-2T} \langle \xi_1 \rangle^{-s+a_1} \\ v_1(\xi_1) \cdots \langle \xi_N \rangle^{-s+a_N} v_N(\xi_N) \langle \eta_1 \rangle^{-s+a_1} \bar{v}_1(\eta_1) \cdots \langle \eta_N \rangle^{-s+a_N} \bar{v}_N(\eta_N).\end{gathered} \tag{16.13}$$

This is really an integral over $\mathbb{R}^{2N-1}$ with respect to Lebesgue measure. Applying Cauchy-Schwarz inequality the absolute value is estimated by

$$\int_{\{(\xi,\eta) \in \mathbb{R}^{2N}; \sum\limits_i \xi_i = \sum\limits_i \eta_i\}} \prod_{i=1}^{N} |v_i(\xi_i)|^2 \langle \sum_l \eta_l \rangle^{2s-2T} \prod_{i=1}^{N} \langle \eta_i \rangle^{-2s+2a_i} \tag{16.14}$$

The domain of integration, given by $\sum\limits_i \eta_i = \sum\limits_i \xi_i$, is covered by the finite number of subsets Γ_j on which in addition $|\eta_j| \geq |\eta_i|$, for all i.

On this set we may take the variables of integration to be η_i for $i \neq j$ and the ξ_l. Then $|\eta_i| \geq |\sum_l \eta_l|/N$ so the second part of the integrand in (16.14) is estimated by

(16.15)
$$\langle \eta_j \rangle^{-2s+2a_j} \langle \sum_l \eta_l \rangle^{2s-2T} \prod_{i \neq j} \langle \eta_i \rangle^{-2s+2a_i} \leq C_N \langle \eta_j \rangle^{-2T+2a_j} \prod_{i \neq j} \langle \eta_i \rangle^{-2s+2a_i} \leq C'_N \prod_{i \neq j} \langle \eta_i \rangle^{-2s}$$

Thus the integral in (16.14) is finite and the desired estimate follows. □

PROPOSITION 16.3. *If* $F \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R})$ *and* $u \in H^s(\mathbb{R}^n)$ *for* $s > \frac{n}{2}$ *an integer then*

$$F(z, u(z)) \in H^s_{\mathrm{loc}}(\mathbb{R}^n). \tag{16.16}$$

PROOF. Since the result is local on $\mathbb{R}^n$ we may multiply by a compactly supported function of z. In fact since $u \in H^s(\mathbb{R}^n)$ is bounded we also multiply by a compactly supported function in $\mathbb{R}$ without changing the result. Thus it suffices to show that

$$F \in \mathcal{C}^\infty_c(\mathbb{R}^n \times \mathbb{R}) \Longrightarrow F(z, u(z)) \in H^s(\mathbb{R}^n). \tag{16.17}$$

Now, Lemma 16.2 can be applied to show that $F(z, u(z)) \in H^s(\mathbb{R}^n)$. Certainly $F(z, u(z)) \in L^2(\mathbb{R}^n)$ since it is continuous and has compact support. Moreover, differentiating s times and applying the chain rule gives

$$D^\alpha F(z, u(z)) = \sum F_{\alpha_1, \dots, \alpha_N}(z, u(z)) D^{\alpha_1} u \cdots D^{\alpha_N} u \tag{16.18}$$

where the sum is over all (finitely many) decomposition with $\sum_{i=1}^N \alpha_i \leq \alpha$ and the $F_\cdot(z, u)$ are smooth with compact support, being various derivitives of $F(z, u)$. Thus it follows from Lemma 16.2 that all terms on the right are in $L^2(\mathbb{R}^n)$ for $|\alpha| \leq s$. □

Note that slightly more sophisticated versions of these arguments give the full result (16.1) but Proposition 16.3 suffices for our purposes below.

17. Some bounded operators

LEMMA 17.1. *If* $J \in \mathcal{C}^k(\Omega^2)$ *is properly supported then the operator with kernel* J *(also denoted* J*) is a map*

$$J : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^k_{\mathrm{loc}}(\Omega) \ \forall \ s \geq -k. \tag{17.1}$$