

5. HILBERT SPACE

We have shown that $L^p(X, \mu)$ is a Banach space – a complete normed space. I shall next discuss the class of Hilbert spaces, a special class of Banach spaces, of which $L^2(X, \mu)$ is a standard example, in which the norm arises from an inner product, just as it does in Euclidean space.

An inner product on a vector space V over \mathbb{C} (one can do the real case too, not much changes) is a *sesquilinear* form

$$V \times V \rightarrow \mathbb{C}$$

written (u, v) , if $u, v \in V$. The ‘sesqui-’ part is just linearity in the first variable

$$(5.1) \quad (a_1 u_1 + a_2 u_2, v) = a_1 (u_1, v) + a_2 (u_2, v),$$

anti-linearly in the second

$$(5.2) \quad (u, a_1 v_1 + a_2 v_2) = \bar{a}_1 (u, v_1) + \bar{a}_2 (u, v_2)$$

and the conjugacy condition

$$(5.3) \quad (u, v) = \overline{(v, u)}.$$

Notice that (5.2) follows from (5.1) and (5.3). If we assume in addition the positivity condition⁸

$$(5.4) \quad (u, u) \geq 0, \quad (u, u) = 0 \Rightarrow u = 0,$$

then

$$(5.5) \quad \|u\| = (u, u)^{1/2}$$

is a *norm* on V , as we shall see.

Suppose that $u, v \in V$ have $\|u\| = \|v\| = 1$. Then $(u, v) = e^{i\theta} |(u, v)|$ for some $\theta \in \mathbb{R}$. By choice of θ , $e^{-i\theta} (u, v) = |(u, v)|$ is real, so expanding out using linearity for $s \in \mathbb{R}$,

$$\begin{aligned} 0 &\leq (e^{-i\theta} u - sv, e^{-i\theta} u - sv) \\ &= \|u\|^2 - 2s \operatorname{Re} e^{-i\theta} (u, v) + s^2 \|v\|^2 = 1 - 2s |(u, v)| + s^2. \end{aligned}$$

The minimum of this occurs when $s = |(u, v)|$ and this is negative unless $|(u, v)| \leq 1$. Using linearity, and checking the trivial cases $u = 0$ or $v = 0$ shows that

$$(5.6) \quad |(u, v)| \leq \|u\| \|v\|, \quad \forall u, v \in V.$$

This is called Schwarz⁹ inequality.

⁸Notice that (u, u) is real by (5.3).

⁹No ‘t’ in this Schwarz.

Using Schwarz' inequality

$$\begin{aligned}\|u + v\|^2 &= \|u\|^2 + (u, v) + (v, u) + \|v\|^2 \\ &\leq (\|u\| + \|v\|)^2 \\ \implies \|u + v\| &\leq \|u\| + \|v\| \quad \forall u, v \in V\end{aligned}$$

which is the triangle inequality.

Definition 5.1. A Hilbert space is a vector space V with an inner product satisfying (5.1) - (5.4) which is complete as a normed space (i.e., is a Banach space).

Thus we have already shown $L^2(X, \mu)$ to be a Hilbert space for any positive measure μ . The inner product is

$$(5.7) \quad (f, g) = \int_X f \bar{g} d\mu,$$

since then (5.3) gives $\|f\|_2$.

Another important identity valid in any inner product spaces is the parallelogram law:

$$(5.8) \quad \|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

This can be used to prove the basic 'existence theorem' in Hilbert space theory.

Lemma 5.2. Let $C \subset H$, in a Hilbert space, be closed and convex (i.e., $su + (1 - s)v \in C$ if $u, v \in C$ and $0 < s < 1$). Then C contains a unique element of smallest norm.

Proof. We can certainly choose a sequence $u_n \in C$ such that

$$\|u_n\| \rightarrow \delta = \inf \{\|v\|; v \in C\}.$$

By the parallelogram law,

$$\begin{aligned}\|u_n - u_m\|^2 &= 2\|u_n\|^2 + 2\|u_m\|^2 - \|u_n + u_m\|^2 \\ &\leq 2(\|u_n\|^2 + \|u_m\|^2) - 4\delta^2\end{aligned}$$

where we use the fact that $(u_n + u_m)/2 \in C$ so must have norm at least δ . Thus $\{u_n\}$ is a Cauchy sequence, hence convergent by the assumed completeness of H . Thus $\lim u_n = u \in C$ (since it is assumed closed) and by the triangle inequality

$$|\|u_n\| - \|u\|| \leq \|u_n - u\| \rightarrow 0$$

So $\|u\| = \delta$. Uniqueness of u follows again from the parallelogram law which shows that if $\|u'\| = \delta$ then

$$\|u - u'\| \leq 2\delta^2 - 4\|(u + u')/2\|^2 \leq 0.$$

□

The fundamental fact about a Hilbert space is that each element $v \in H$ defines a continuous linear functional by

$$H \ni u \longmapsto (u, v) \in \mathbb{C}$$

and conversely *every* continuous linear functional arises this way. This is also called the Riesz representation theorem.

Proposition 5.3. *If $L : H \rightarrow \mathbb{C}$ is a continuous linear functional on a Hilbert space then there is a unique element $v \in H$ such that*

$$(5.9) \quad Lu = (u, v) \quad \forall u \in H,$$

Proof. Consider the linear space

$$M = \{u \in H ; Lu = 0\}$$

the null space of L , a continuous linear functional on H . By the assumed continuity, M is closed. We can suppose that L is *not* identically zero (since then $v = 0$ in (5.9)). Thus there exists $w \notin M$. Consider

$$w + M = \{v \in H ; v = w + u, u \in M\}.$$

This is a closed convex subset of H . Applying Lemma 5.2 it has a unique smallest element, $v \in w + M$. Since v minimizes the norm on $w + M$,

$$\|v + su\|^2 = \|v\|^2 + 2 \operatorname{Re}(su, v) + \|s\|^2 \|u\|^2$$

is stationary at $s = 0$. Thus $\operatorname{Re}(u, v) = 0 \quad \forall u \in M$, and the same argument with s replaced by is shows that $(v, u) = 0 \quad \forall u \in M$.

Now $v \in w + M$, so $Lv = Lw \neq 0$. Consider the element $w' = w/Lw \in H$. Since $Lw' = 1$, for any $u \in H$

$$L(u - (Lu)w') = Lu - Lu = 0.$$

It follows that $u - (Lu)w' \in M$ so if $w'' = w'/\|w'\|^2$

$$(u, w'') = ((Lu)w', w'') = Lu \frac{(w', w')}{\|w'\|^2} = Lu.$$

The uniqueness of v follows from the positivity of the norm. □

Corollary 5.4. *For any positive measure μ , any continuous linear functional*

$$L : L^2(X, \mu) \rightarrow \mathbb{C}$$

is of the form

$$Lf = \int_X f \bar{g} d\mu, \quad g \in L^2(X, \mu).$$

Notice the apparent power of ‘abstract reasoning’ here! Although we seem to have constructed g out of nowhere, its existence follows from the *completeness* of $L^2(X, \mu)$, but it is very convenient to express the argument abstractly for a general Hilbert space.