1. Continuous functions

A the beginning I want to remind you of things I think you already know and then go on to show the direction the course will be taking. Let me first try to set the context.

One basic notion I assume you are reasonably familiar with is that of a *metric space* ([5] p.9). This consists of a set, X, and a distance function

$$d : X \times X = X^2 \longrightarrow [0, \infty) ,$$

satisfying the following three axioms:

$$\begin{aligned} &i)\ d(x, y) = 0 \Leftrightarrow x = y,\ (\text{and } d(x, y) \geq 0)\\ &ii)\ d(x, y) = d(y, x)\ \forall\ x, y \in X \\ &iii)\ d(x, y) \leq d(x, z) + d(z, y)\ \forall\ x, y, z \in X. \end{aligned} \tag{1.1}$$

The basic theory of metric spaces deals with properties of subsets (open, closed, compact, connected), sequences (convergent, Cauchy) and maps (continuous) and the relationship between these notions. Let me just remind you of one such result.

Proposition 1.1. *A map $f : X \to Y$ between metric spaces is continuous if and only if one of the three following equivalent conditions holds*

(1) $f^{-1}(O) \subset X$ *is open* $\forall\ O \subset Y$ *open.*
(2) $f^{-1}(C) \subset X$ *is closed* $\forall\ C \subset Y$ *closed.*
(3) $\lim_{n\to\infty} f(x_n) = f(x)$ *in* Y *if* $x_n \to x$ *in* X.

The basic example of a metric space is Euclidean space. Real n-dimensional Euclidean space, $\mathbb{R}^n$, is the set of ordered n-tuples of real numbers

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n ,\ x_j \in \mathbb{R} ,\ j = 1, \dots, n .$$

It is also the basic example of a vector (or linear) space with the operations

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ cx &= (cx_1, \dots, cx_n) . \end{aligned}$$

The metric is usually taken to be given by the Euclidean metric

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2} = (\sum_{j=1}^{n} x_j^2)^{1/2} ,$$

in the sense that

$$d(x, y) = |x - y| .$$

Let us abstract this immediately to the notion of a normed vector space, or normed space. This is a vector space V (over $\mathbb{R}$ or $\mathbb{C}$) equipped with a *norm*, which is to say a function

$$\| \ \| : V \longrightarrow [0, \infty)$$

satisfying

$$\begin{aligned} &i)\ \|v\| = 0 \Longleftrightarrow v = 0, \\ &ii)\ \|cv\| = |c|\,\|v\| \ \forall\ c \in \mathbb{K}, \\ &iii)\ \|v + w\| \le \|v\| + \|w\|. \end{aligned} \tag{1.2}$$

This means that (V, d), $d(v, w) = \|v - w\|$ is a vector space; I am also using $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$ as is appropriate.

The case of a finite dimensional normed space is not very interesting because, apart from the dimension, they are all "the same". We shall say (in general) that two norms $\| \bullet \|_1$ and $\| \bullet \|_2$ on V are *equivalent* of there exists $C > 0$ such that

$$\frac{1}{C}\|v\|_1 \le \|v\|_2 \le C\|v\|_1 \ \forall\ v \in V\,.$$

Proposition 1.2. *Any two norms on a finite dimensional vector space are equivalent.*

So, we are mainly interested in the infinite dimensional case. I will start the course, in a slightly unorthodox manner, by concentrating on one such normed space (really one class). Let X be a metric space. The case of a continuous function, $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is a special case of Proposition 1.1 above. We then define

$$C(X) = \{f : X \to \mathbb{R},\ f \text{ bounded and continuous}\}\,.$$

In fact the same notation is generally used for the space of complex-valued functions. If we want to distinguish between these two possibilities we can use the more pedantic notation $C(X; \mathbb{R})$ and $C(X; \mathbb{C})$. Now, the 'obvious' norm on this linear space is the supremum (or 'uniform') norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|\ .$$

Here X is an arbitrary metric space. For the moment X is supposed to be a "physical" space, something like $\mathbb{R}^n$. Corresponding to the finite-dimensionality of $\mathbb{R}^n$ we often assume (or demand) that X is *locally compact*. This just means that every point has a compact neighborhood, i.e., is in the interior of a compact set. Whether locally compact or not we can consider

$$\mathcal{C}_0(X) = \left\{ f \in \mathcal{C}(X); \forall\ \epsilon > 0\ \exists\ K \Subset X s.t. \sup_{x \notin K} |f(x)| \le \epsilon \right\}. \tag{1.3}$$

Here the notation $K \Subset X$ means 'K is a compact subset of X'.

If V is a normed linear space we are particularly interested in the continuous linear functionals on V. Here 'functional' just means function but V is allowed to be 'large' (not like $\mathbb{R}^n$) so 'functional' is used for historical reasons.

Proposition 1.3. *The following are equivalent conditions on a linear functional $u : V \longrightarrow \mathbb{R}$ on a normed space V.*

(1) *u is continuous.*
(2) *u is continuous at 0.*
(3) *$\{u(f) \in \mathbb{R}\,;\ f \in V\,,\ \|f\| \le 1\}$ is bounded.*
(4) *$\exists\ C$ s.t. $|u(f)| \le C\|f\|\ \forall\ f \in V$.*

Proof. (1) $\Longrightarrow$ (2) by definition. Then (2) implies that $u^{-1}(-1, 1)$ is a neighborhood of $0 \in V$, so for some $\epsilon > 0$, $u(\{f \in V; \|f\| < \epsilon\}) \subset (-1, 1)$. By linearity of u, $u(\{f \in V; \|f\| < 1\}) \subset (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ is bounded, so (2) $\Longrightarrow$ (3). Then (3) implies that

$$|u(f)| \le C\ \forall\ f \in V, \|f\| \le 1$$

for some C. Again using linearity of u, if $f \ne 0$,

$$|u(f)| \le \|f\| u\left(\frac{f}{\|f\|}\right) \le C\|f\|\,,$$

giving (4). Finally, assuming (4),

$$|u(f) - u(g)| = |u(f - g)| \le C\|f - g\|$$

shows that u is continuous at any point $g \in V$. □

In view of this identification, continuous linear functionals are often said to be *bounded*. One of the important ideas that we shall exploit later is that of 'duality'. In particular this suggests that it is a good idea to examine the totality of bounded linear functionals on V. The *dual* space is

$$V' = V^* = \{u : V \longrightarrow \mathbb{K}\,,\ \text{linear and bounded}\}\,.$$

This is also a normed linear space where the linear operations are

$$\begin{matrix} (u + v)(f) = u(f) + v(f) \\ (cu)(f) = c(u(f)) \end{matrix}\ \forall\ f \in V. \tag{1.4}$$

The natural norm on V' is

$$\|u\| = \sup_{\|f\| \le 1} |u(f)|.$$

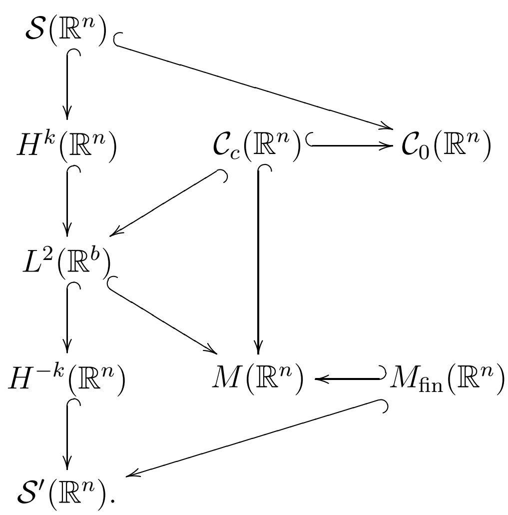
This is just the 'best constant' in the boundedness estimate,

$$\|u\| = \inf \{C; |u(f)| \le C\|f\| \ \forall\ f \subset V\}\ .$$

One of the basic questions I wish to pursue in the first part of the course is: What is the dual of $\mathcal{C}_0(X)$ for a locally compact metric space X? The answer is given by Riesz' representation theorem, in terms of (Borel) measures.

Let me give you a vague picture of 'regularity of functions' which is what this course is about, even though I have not introduced most of these spaces yet. Smooth functions (and small spaces) are towards the top. Duality flips up and down and as we shall see L^2, the space of Lebesgue square-integrable functions, is generally 'in the middle'. What I will discuss first is the right side of the diagramme, where we have the space of continuous functions on $\mathbb{R}^n$ which vanish at infinity and its dual space, $M_{\text{fin}}(\mathbb{R}^n)$, the space of finite Borel measures. There are many other spaces that you may encounter, here I only include test functions, Schwartz functions, Sobolev spaces and their duals; k is a general positive integer.

(1.5)



I have set the goal of understanding the dual space $M_{\text{fin}}(\mathbb{R}^n)$ of $\mathcal{C}_0(X)$, where X is a locally compact metric space. This will force me to go through the elements of measure theory and Lebesgue integration. It does require a little forcing!

The basic case of interest is $\mathbb{R}^n$. Then an obvious example of a continuous linear functional on $\mathcal{C}_0(\mathbb{R}^n)$ is given by Riemann integration, for instance over the unit cube $[0,1]^n$:

$$u(f)=\int_{[0,1]^n} f(x)\,dx\,.$$

In some sense we must show that *all* continuous linear functionals on $\mathcal{C}_0(X)$ are given by integration. However, we have to interpret integration somewhat widely since there are also *evaluation functionals*. If $z\in X$ consider the Dirac delta

$$\delta_z(f)=f(z)\,.$$

This is also called a *point mass* of z. So we need a theory of measure and integration wide enough to include both of these cases.

One special feature of $\mathcal{C}_0(X)$, compared to general normed spaces, is that there is a notion of positivity for its elements. Thus $f\geq 0$ just means $f(x)\geq 0\ \forall\ x\in X$.

Lemma 1.4. *Each $f\in\mathcal{C}_0(X)$ can be decomposed uniquely as the difference of its positive and negative parts*

$$f=f_+-f_-\,,\ f_\pm\in\mathcal{C}_0(X)\,,\ f_\pm(x)\leq|f(x)|\ \ \forall\ x\in X\,. \tag{1.6}$$

Proof. Simply define

$$f_\pm(x)=\begin{cases}\pm f(x) & \text{if}\quad \pm f(x)\geq 0\\ 0 & \text{if}\quad \pm f(x)<0\end{cases}$$

for the same sign throughout. Then (1.6) holds. Observe that f_+ is continuous at each $y\in X$ since, with U an appropriate neighborhood of y, in each case

$$f(y)>0\Longrightarrow f(x)>0 \text{ for } x\in U\Longrightarrow f_+=f \text{ in } U$$

$$f(y)<0\Longrightarrow f(x)<0 \text{ for } x\in U\Longrightarrow f_+=0 \text{ in } U$$

$$f(y)=0\Longrightarrow \text{ given }\ \epsilon>0\ \exists\ U \text{ s.t. }\ |f(x)|<\epsilon \text{ in } U$$

$$\Longrightarrow |f_+(x)|<\epsilon \text{ in } U\,.$$

Thus $f_-=f-f_+\in\mathcal{C}_0(X)$, since both f_+ and f_- vanish at infinity. □

We can similarly split elements of the dual space into positive and negative parts although it is a little bit more delicate. We say that $u\in(\mathcal{C}_0(X))'$ is positive if

$$u(f)\geq 0\ \forall\ 0\leq f\in\mathcal{C}_0(X)\,. \tag{1.7}$$

For a general (real) $u\in(\mathcal{C}_0(X))'$ and for each $0\leq f\in\mathcal{C}_0(X)$ set

$$u_+(f)=\sup\left\{u(g)\,;\,g\in\mathcal{C}_0(X)\,,\ 0\leq g(x)\leq f(x)\ \forall\ x\in X\right\}\,. \tag{1.8}$$

This is certainly finite since $u(g) \le C\|g\|_\infty \le C\|f\|_\infty$. Moreover, if $0 < c \in \mathbb{R}$ then $u_+(cf) = cu_+(f)$ by inspection. Suppose $0 \le f_i \in \mathcal{C}_0(X)$ for $i = 1, 2$. Then given $\epsilon > 0$ there exist $g_i \in \mathcal{C}_0(X)$ with $0 \le g_i(x) \le f_i(x)$ and

$$u_+(f_i) \le u(g_i) + \epsilon\,.$$

It follows that $0 \le g(x) \le f_1(x) + f_2(x)$ if $g = g_1 + g_2$ so

$$u_+(f_1 + f_2) \ge u(g) = u(g_1) + u(g_2) \ge u_+(f_1) + u_+(f_2) - 2\epsilon\,.$$

Thus

$$u_+(f_1 + f_2) \ge u_+(f_1) + u_+(f_2).$$

Conversely, if $0 \le g(x) \le f_1(x) + f_2(x)$ set $g_1(x) = \min(g, f_1) \in \mathcal{C}_0(X)$ and $g_2 = g - g_1$. Then $0 \le g_i \le f_i$ and $u_+(f_1) + u_+(f_2) \ge u(g_1) + u(g_2) = u(g)$. Taking the supremum over g, $u_+(f_1 + f_2) \le u_+(f_1) + u_+(f_2)$, so we find

$$u_+(f_1 + f_2) = u_+(f_1) + u_+(f_2)\,. \tag{1.9}$$

Having shown this effective linearity on the positive functions we can obtain a linear functional by setting

$$u_+(f) = u_+(f_+) - u_+(f_-)\ \forall\ f \in \mathcal{C}_0(X)\,. \tag{1.10}$$

Note that (1.9) shows that $u_+(f) = u_+(f_1) - u_+(f_2)$ for any decomposiiton of $f = f_1 - f_2$ with $f_i \in \mathcal{C}_0(X)$, both positive. [Since $f_1 + f_- = f_2 + f_+$ so $u_+(f_1) + u_+(f_-) = u_+(f_2) + u_+(f_+)$.] Moreover,

$$\begin{aligned} |u_+(f)| \le \max(u_+(f_+), u(f_-)) &\le \|u\|\,\|f\|_\infty \\ &\Longrightarrow \|u_+\| \le \|u\|\,. \end{aligned}$$

The functional

$$u_- = u_+ - u$$

is also positive, since $u_+(f) \ge u(f)$ for all $0 \le f \in \mathcal{C}_0(x)$. Thus we have proved

Lemma 1.5. *Any element $u \in (\mathcal{C}_0(X))'$ can be decomposed,*

$$u = u_+ - u_-$$

into the difference of positive elements with

$$\|u_+\|\,,\ \|u_-\| \le \|u\|\,.$$

The idea behind the definition of u_+ is that u itself is, more or less, "integration against a function" (even though we do *not* know how to interpret this yet). In defining u_+ from u we are effectively throwing away the negative part of that 'function.' The next step is to show that a positive functional corresponds to a 'measure' meaning a function measuring the size of sets. To define this we really want to evaluate u on the characteristic function of a set

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E\,. \end{cases}$$

The problem is that χ_E is not continuous. Instead we use an idea similar to (1.8).

If $0 \le u \in (\mathcal{C}_0(X))'$ and $U \subset X$ is *open,* set[1]

$$\mu(U) = \sup\left\{u(f)\,;\, 0 \le f(x) \le 1, f \in \mathcal{C}_0(X)\,,\, \operatorname{supp}(f) \Subset U\right\}\,. \tag{1.11}$$

Here the support of f, $\operatorname{supp}(f)$, is the *closure* of the set of points where $f(x) \neq 0$. Thus $\operatorname{supp}(f)$ is always closed, in (1.11) we only admit f if its support is a compact subset of U. The reason for this is that, only then do we 'really know' that $f \in \mathcal{C}_0(X)$.

Suppose we try to measure general sets in this way. We can do this by defining

$$\mu^*(E) = \inf\left\{\mu(U)\,;\, U \supset E\,,\, U \text{ open}\right\}\,. \tag{1.12}$$

Already with μ it may happen that $\mu(U) = \infty$, so we think of

$$\mu^* : \mathcal{P}(X) \to [0, \infty] \tag{1.13}$$

as defined on the *power set* of X and taking values in the extended positive real numbers.

Definition 1.6. *A positive extended function, μ^*, defined on the power set of X is called an* outer measure *if $\mu^*(\emptyset) = 0$, $\mu^*(A) \le \mu^*(B)$ whenever $A \subset B$ and*

$$\mu^*(\bigcup_j A_j) \le \sum_j \mu(A_j)\ \forall \quad \{A_j\}_{j=1}^\infty \subset \mathcal{P}(X)\,. \tag{1.14}$$

Lemma 1.7. *If u is a positive continuous linear functional on $\mathcal{C}_0(X)$ then μ^*, defined by* (1.11), (1.12) *is an outer measure.*

To prove this we need to find enough continuous functions. I have relegated the proof of the following result to Problem 2.

Lemma 1.8. *Suppose U_i, $i = 1, \dots, N$ is ,a finite collection of open sets in a locally compact metric space and $K \Subset \bigcup_{i=1}^N U_i$ is a compact subset, then there exist continuous functions $f_i \in C(X)$ with $0 \le f_i \le 1$,* $\operatorname{supp}(f_i) \Subset U_i$ *and*

$$\sum_i f_i = 1 \text{ in a neighborhood of } K\,. \tag{1.15}$$

[1] See [5] starting p.42 or [1] starting p.206.

Proof of Lemma 1.7. We have to prove (1.14). Suppose first that the A_i are open, then so is $A = \bigcup_i A_i$. If $f \in C(X)$ and $\operatorname{supp}(f) \Subset A$ then $\operatorname{supp}(f)$ is covered by a finite union of the A_is. Applying Lemma 1.8 we can find f_i's, all but a finite number identically zero, so $\operatorname{supp}(f_i) \Subset A_i$ and $\sum_i f_i = 1$ in a neighborhood of $\operatorname{supp}(f)$.

Since $f = \sum_i f_i f$ we conclude that

$$u(f) = \sum_i u(f_i f) \Longrightarrow \mu^*(A) \le \sum_i \mu^*(A_i)$$

since $0 \le f_i f \le 1$ and $\operatorname{supp}(f_i f) \Subset A_i$.

Thus (1.14) holds when the A_i are open. In the general case if $A_i \subset B_i$ with the B_i open then, from the definition,

$$\mu^*(\bigcup_i A_i) \le \mu^*(\bigcup_i B_i) \le \sum_i \mu^*(B_i)\,.$$

Taking the infimum over the B_i gives (1.14) in general. □