

**FOURTH PROBLEM SET FOR 18.155
DUE NOVEMBER 7 IN CLASS OR 2-174.**

Especially for undergraduates I will accept just one of these problems as enough work for the week. Notice that the first problem is considered the deepest result in ‘elementary’ distribution theory.

- Problem 1* (Schwartz kernel theorem). (1) Show (mostly recalling things from class) that the topology of $\mathcal{S}(\mathbb{R}^n)$ can be expressed as the projective limit of the Hilbert topologies on the weighted Sobolev spaces $\langle x \rangle^{-k} H^k(\mathbb{R}^n)$ as $k \rightarrow \infty$
- (2) Conclude that the dual space $\mathcal{S}'(\mathbb{R}^n)$ can be topologized as the inductive limit of the weighted Sobolev spaces $\langle x \rangle^k H^{-k}(\mathbb{R}^n)$ as $k \rightarrow \infty$.
- (3) Show that the Fourier transform, multiplication by $\langle x \rangle^m$ for any m and the map $\langle D \rangle^m$ (defined as the preceding map conjugated by the Fourier transform) are all isomorphisms of both $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.
- (4) Show that for a linear map (an operator)

$$(0.1) \quad A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

continuity with respect to these topologies is equivalent to the existence of some k such that A extends by continuity (in the Hilbert norms) to $A : \langle x \rangle^{-k} H^k(\mathbb{R}^n) \longrightarrow \langle x \rangle^k H^{-k}(\mathbb{R}^n)$.

- (5) Show how to compose A on the left and right with maps from (3) and arrive at a continuous linear map A' from $H^{-n}(\mathbb{R}^n)$ to $H^n(\mathbb{R}^n)$.
- (6) Recall the Sobolev embedding theorem and that the delta ‘function’ at any point is in $H^{-n}(\mathbb{R}^n)$ and use this to conclude that the formula $a(x, y) = (A'(\delta_y))(x)$ defines a continuous bounded function on \mathbb{R}^{2n} .
- (7) Show that the map

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n}), (\phi, \psi) \longmapsto \phi \boxtimes \psi(x, y) = \phi(x)\psi(y)$$

is jointly continuous (i.e. is continuous in the product metric topology).

- (8) Show that if $\beta \in \mathcal{S}'(\mathbb{R}^{2n})$ then the formula

$$(0.2) \quad (B\psi)\phi = \beta(\phi \boxtimes \psi) \quad \forall \phi, \psi \in \mathcal{S}(\mathbb{R}^n)$$

defines a continuous linear map $B : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$.

- (9) Going back to (6) show that a thought of as a distribution defines A' in this way.
- (10) Conclude that every continuous linear operator (0.1) arises from the construction (0.2).
- (11) If you get this far and still have the energy, show that conversely β is determined by B through (0.2) and finally arrive at Schwartz kernel theorem:
- (12)

Theorem 0.1. *There is a 1-1 correspondence between continuous linear maps*

$$(0.3) \quad A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{n'})$$

and elements of $\mathcal{S}'(\mathbb{R}^{n+n'})$.

Problem 2. Work out the elementary behavior of the heat equation.

i) Show that the function on $\mathbb{R} \times \mathbb{R}^n$, for $n \geq 1$,

$$F(t, x) = \begin{cases} t^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is measurable, bounded on the any set $\{|(t, x)| \geq R\}$ and is integrable on $\{|(t, x)| \leq R\}$ for any $R > 0$.

- ii) Conclude that F defines a tempered distribution on \mathbb{R}^{n+1} .
 iii) Show that F is C^∞ outside the origin.
 iv) Show that F satisfies the heat equation

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t, x) = 0 \text{ in } (t, x) \neq 0.$$

v) Show that F satisfies

$$(0.4) \quad F(s^2t, sx) = s^{-n}F(t, x) \text{ in } \mathcal{S}'(\mathbb{R}^{n+1})$$

where the left hand side is defined by duality " $F(s^2t, sx) = F_s$ " where

$$F_s(\phi) = s^{-n-2}F(\phi_{1/s}), \quad \phi_{1/s}(t, x) = \phi\left(\frac{t}{s^2}, \frac{x}{s}\right).$$

vi) Conclude that

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t, x) = G(t, x)$$

where $G(t, x)$ satisfies

$$(0.5) \quad G(s^2t, sx) = s^{-n-2}G(t, x) \text{ in } \mathcal{S}'(\mathbb{R}^{n+1})$$

in the same sense as above and has support at most $\{0\}$.

vii) Hence deduce that

$$(0.6) \quad (\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t, x) = c\delta(t)\delta(x)$$

for some real constant c .

Hint: Check which distributions with support at $(0, 0)$ satisfy (0.5).

viii) If $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ show that $u = F \star \psi$ satisfies

$$(0.7) \quad u \in \mathcal{C}^\infty(\mathbb{R}^{n+1}) \text{ and}$$

$$\sup_{x \in \mathbb{R}^n, t \in [-S, S]} (1 + |x|)^N |D^\alpha u(t, x)| < \infty \quad \forall S > 0, \alpha \in \mathbb{N}^{n+1}, N.$$

ix) Supposing that u satisfies (0.7) and is a real-valued solution of

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)u(t, x) = 0$$

in \mathbb{R}^{n+1} , show that

$$v(t) = \int_{\mathbb{R}^n} u^2(t, x)$$

is a non-increasing function of t .

Hint: Multiply the equation by u and integrate over a slab $[t_1, t_2] \times \mathbb{R}^n$.

- x) Show that c in (0.6) is non-zero by arriving at a contradiction from the assumption that it is zero. Namely, show that if $c = 0$ then u in viii) satisfies the conditions of ix) and also vanishes in $t < T$ for some T (depending on ψ). Conclude that $u = 0$ for all ψ . Using properties of convolution show that this in turn implies that $F = 0$ which is a contradiction.
- xi) So, finally, we know that $E = \frac{1}{c}F$ is a fundamental solution of the heat operator which vanishes in $t < 0$. Explain why this allows us to show that for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ there is a solution of

$$(0.8) \quad (\partial_t - \sum_{j=1}^n \partial_{x_j}^2)u = \psi, \quad u = 0 \text{ in } t < T \text{ for some } T.$$

What is the largest value of T for which this holds?

- xii) Can you give a heuristic, or indeed a rigorous, explanation of why

$$c = \int_{\mathbb{R}^n} \exp\left(-\frac{|x|^2}{4}\right) dx?$$