THIRD (DELAYED) PROBLEM SET FOR 18.155
DUE OCTOBER 17 IN CLASS OR 2-174.

These are Problems 24 - 30 of the notes on the web.

Problem 1. Prove the continuity in the mean of $L^2$ functions on $\mathbb{R}^n$, that 
\[ \sup_{|t|<\epsilon} \int_{\mathbb{R}^n} |u(x+t) - u(x)|^2dx \to 0 \text{ as } \epsilon \to 0. \]

You will probably have to go back to first principles to do this. Show that it is enough to assume $u \geq 0$ has compact support. Then show it is enough to assume that $u$ is a simple, and integrable, function. Finally look at the definition of Lebesgue measure and show that if $E \subset \mathbb{R}^n$ is Borel and has finite Lebesgue measure then 
\[ \lim_{|t| \to \infty} \mu(E \setminus (E + t)) = 0 \]
where $\mu$ = Lebesgue measure and 
\[ E + t = \{ p \in \mathbb{R}^n ; p' + t , p' \in E \} . \]

Problem 2. Prove Leibniz' formula 
\[ D^\alpha x (\phi \psi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta x \phi \cdot D^{\alpha-\beta} x \psi \]
for any $C^\infty$ functions and $\phi$ and $\psi$. Here $\alpha$ and $\beta$ are multiindices, $\beta \leq \alpha$ means $\beta_j \leq \alpha_j$ for each $j$ and 
\[ \binom{\alpha}{\beta} = \prod_j \binom{\alpha_j}{\beta_j} . \]

I suggest induction!

Problem 3. Prove the generalization of a result from class that $u \in S'(\mathbb{R}^n)$, supp$u$ $\subset \{0\}$ (which means that $u(\phi) = 0$ for all $\phi \in S(\mathbb{R}^n)$ such that $\phi = 0$ in $|x| < \epsilon$ for some $\epsilon > 0$) implies there are constants $c_\alpha$ for $|\alpha| \leq m$, for some $m$, such that 
\[ u = \sum_{|\alpha| \leq m} c_\alpha D^\alpha \delta . \]

Hint This is not so easy! I would be happy if you can show that $u \in M(\mathbb{R}^n)$, supp$u$ $\subset \{0\}$ implies $u = c\delta$. To see this, you can show that 
\[ \phi \in \mathcal{S}(\mathbb{R}^n), \phi(0) = 0 \]
\[ \Rightarrow \exists \varphi_j \in \mathcal{S}(\mathbb{R}^n), \varphi_j(x) = 0 \text{ in } |x| \leq \epsilon_j \text{ where } \epsilon_j \downarrow 0 \]
and \[ \sup |\varphi_j - \phi| \to 0 \text{ as } j \to \infty . \]

To prove the general case you need something similar — that given $m$, if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $D^\alpha \varphi(0) = 0$ for $|\alpha| \leq m$ then $\exists \varphi_j \in \mathcal{S}(\mathbb{R}^n)$, $\varphi_j = 0$ in $|x| \leq \epsilon_j \text{, } \epsilon_j \downarrow 0$ such that $\varphi_j \to \varphi$ in the $C^m$ norm.
Problem 4. If $m \in \mathbb{N}$, $m' > 0$ show that $u \in H^m(\mathbb{R}^n)$ and $D^\alpha u \in H^{m'}(\mathbb{R}^n)$ for all $|\alpha| \leq m$ implies $u \in H^{m+m'}(\mathbb{R}^n)$. Is the converse true?

Problem 5. Show that every element $u \in L^2(\mathbb{R}^n)$ can be written as a sum
\[
    u = u_0 + \sum_{j=1}^{n} D_j u_j, \quad u_j \in H^1(\mathbb{R}^n), \quad j = 0, \ldots, n.
\]

Problem 6. Consider for $n = 1$, the locally integrable function (the Heaviside function),
\[
    H(x) = \begin{cases} 
        0 & x \leq 0 \\
        1 & x > 0 
    \end{cases}.
\]
Show that $D_x H(x) = c\delta$; what is the constant $c$?

Problem 7. For what range of orders $m$ is it true that $\delta \in H^m(\mathbb{R}^n)$, $\delta(\varphi) = \varphi(0)$?