Problem 1. Wavefront set computations and more – all pretty easy, especially if you use results from class.

i) Compute $\text{WF}(\delta)$ where $\delta \in S' (\mathbb{R}^n)$ is the Dirac delta function at the origin.

ii) Compute $\text{WF}(H(x))$ where $H(x) \in S' (\mathbb{R})$ is the Heaviside function

$$H(x) = \begin{cases} 
1 & x > 0 \\
0 & x \leq 0 
\end{cases}.$$ 

Hint: $D_x$ is elliptic in one dimension, hit $H$ with it.

iii) Compute $\text{WF}(E)$, $E = iH(x_1)\delta(x')$ which is the Heaviside in the first variable on $\mathbb{R}^n$, $n > 1$, and delta in the others.

iv) Show that $D_{x_1} E = \delta$, so $E$ is a fundamental solution of $D_{x_1}$.

v) If $f \in C_{c}^{-\infty} (\mathbb{R}^n)$ show that $u = E \ast f$ solves $D_{x_1} u = f$.

vi) What does our estimate on $\text{WF}(E \ast f)$ tell us about $\text{WF}(u)$ in terms of $\text{WF}(f)$?

Problem 2. The wave equation in two variables (or one spatial variable).

i) Recall that the Riemann function

$$E(t,x) = \begin{cases} 
-\frac{1}{4} & \text{if } t > x \text{ and } t > -x \\
0 & \text{otherwise} 
\end{cases}$$

is a fundamental solution of $D_{t}^2 - D_{x}^2$ (check my constant).

ii) Find the singular support of $E$.

iii) Write the Fourier transform (dual) variables as $\tau, \xi$ and show that

$$\text{WF}(E) \subset \{(t, x, \tau, \xi); x = t > 0 \text{ and } \xi + \tau = 0 \} \cup \{(t, x, \tau, \xi); -x = t > 0 \text{ and } \xi = \tau \}.$$ 

iv) Show that if $f \in C_{c}^{-\infty} (\mathbb{R}^2)$ then $u = E \ast f$ satisfies $(D_{t}^2 - D_{x}^2) u = f$.

v) With $u$ defined as in iv) show that

$$\text{supp}(u) \subset \{(t, x); \exists (t', x') \in \text{supp}(f) \text{ with } t' + x' \leq t + x \text{ and } t' - x' \leq t - x \}.$$ 

vi) Sketch an illustrative example of v).

vii) Show that, still with $u$ given by iv),

$$\text{sing supp}(u) \subset \{(t, x); \exists (t', x') \in \text{sing sup}(f) \text{ with } t \geq t' \text{ and } t + x = t' + x' \text{ or } t - x = t' - x' \}.$$ 

viii) Bound $\text{WF}(u)$ in terms of $\text{WF}(f)$.
Problem 3. A little uniqueness theorems. Suppose \( u \in C^{-\infty}_c(\mathbb{R}^n) \) recall that the Fourier transform \( \hat{u} \in C^\infty(\mathbb{R}^n) \). Now, suppose \( u \in C^{-\infty}_c(\mathbb{R}^n) \) satisfies \( P(D)u = 0 \) for some non-trivial polynomial \( P \), show that \( u = 0 \).