SIXTH ASSIGNMENT, SOLUTIONS
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Problem 1. Hilbert space and the Riesz representation theorem. If you need help with this, it can be found in lots of places – for instance [1] has a nice treatment.

i) A pre-Hilbert space is a vector space V (over $\mathbb{C}$) with a 'positive definite sesquilinear inner product' i.e. a function

$$V \times V \ni (v, w) \longmapsto \langle v, w\rangle \in \mathbb{C}$$

satisfying

- $\langle w, v\rangle = \overline{\langle v, w\rangle}$
- $\langle a_1 v_1 + a_2 v_2, w\rangle = a_1\langle v_1, w\rangle + a_2\langle v_2, w\rangle$
- $\langle v, v\rangle \geq 0$
- $\langle v, v\rangle = 0 \Rightarrow v = 0.$

Prove Schwarz' inequality, that

$$|\langle u, v\rangle| \leq \langle u, u\rangle^{\frac{1}{2}}\langle v, v\rangle^{\frac{1}{2}} \ \forall\ u, v \in V.$$

Hint: Reduce to the case $\langle v, v\rangle = 1$ and then expand

$$\langle u - \langle u, v\rangle v, u - \langle u, v\rangle v\rangle \geq 0.$$

Solution. If $v = 0$ then $\langle u, v\rangle = 0$ and Schwarz' inequality certainly holds. If $v \neq 0$ then $\langle v, v\rangle > 0$ so we can divide through by $\langle v, v\rangle^{\frac{1}{2}}$ and replace v by $\hat{v} = v/\langle v, v\rangle^{\frac{1}{2}}$ which has $\langle \hat{v}, \hat{v}\rangle = 1$. Thus we may as well assume $\langle v, v\rangle = 1$.

Now using the linearity and anti-linearity which follow from the conditions above,

$$\begin{aligned}\langle u - \langle u, v\rangle v, u - \langle u, v\rangle v\rangle \\ &= \langle u, u\rangle - \overline{\langle u, v\rangle}\langle u, v\rangle - \langle u, v\rangle\langle v, u\rangle + |\langle u, v\rangle|^2 \\ &= \langle u, u\rangle - |\langle u, v\rangle|^2 \geq 0.\end{aligned}$$

This proves Schwarz' inequality. □

ii) Show that $\|v\| = \langle v, v\rangle^{1/2}$ is a norm and that it satisfies the parallelogram law:

$$\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 = 2\|v_1\|^2 + 2\|v_2\|^2 \ \forall\ v_1, v_2 \in V. \tag{1}$$

Solution. As with the special case of L^2 that I did in class, the triangle inequality follows from Schwarz' inequality:

$$\|u + v\|^2 = \langle u + v, u + v\rangle \leq \|u\|^2 + 2|\langle u, v\rangle|^2 + \|v\|^2 \leq (\|u\| + \|v\|)^2.$$

The other properties of a norm follow directly. For the parallelogram law expand out the left side:

$$\|v_1+v_2\|^2+\|v_1-v_2\|^2 = \\ \|v_1\|^2+\langle v_1,v_2\rangle+\langle v_2,v_1\rangle+\|v_2\|^2+\|v_1\|^2-\langle v_1,v_2\rangle-\langle v_2,v_1\rangle+\|v_2\|^2 \\ =2\|v_1\|^2+2\|v_2\|^2.$$

□

iii) Conversely, suppose that V is a linear space over $\mathbb{C}$ with a norm which satisfies (1). Show that

$$4\langle v,w\rangle=\|v+w\|^2-\|v-w\|^2+i\|v+iw\|^2-i\|v-iw\|^2$$

defines a pre-Hilbert inner product which gives the original norm.

Solution. With $\langle u,v\rangle$ defined this way

$$4\langle v,v\rangle=4\|v\|^2+i|1+i|\|v\|^2-i|1-i|\|v\|^2=4\|v\|^2$$

and it follows directly that $\langle -u,v\rangle=-\langle u,v\rangle$, $\langle iu,v\rangle=i\langle u,v\rangle$, $\langle c^2u,c^2v\rangle= c^4\langle u,v\rangle$, $c>0$. Furthermore

$$\begin{aligned}4\langle 2u,v\rangle&=\|2u+v\|^2-\|2u-v\|^2+i\|2u+iv\|^2-i\|2u-iv\|^2\\&=2\|u+v\|^2+2\|v\|^2-\|u\|^2-2\|u-v\|^2-2\|v\|^2+\|u\|^2\\&\quad+2i\|u+iv\|^2+2i\|v\|^2-i\|u\|^2-2i|u-iv\|^2-2i\|v\|^2+i\|u\|^2\\&=8\langle u,v\rangle.\end{aligned}$$

Iterating this it follows that $\langle ku,lv\rangle=kl\langle u,v\rangle$ for integers k,l and hence that $\langle au,v\rangle=a\langle u,v\rangle$ for all rational a. The triangle inequality gives continuity so that $\langle zu,v\rangle=z\langle u,v\rangle$ for all $z\in\mathbb{C}$. Addivity in the first variable then also follows from the parallelogram law:

$$\begin{aligned}4\langle u_1+u_2,v\rangle&=\|u_1+u_2+v\|^2-\|u_1+u_2-v\|^2+i\|u_1+u_2+iv\|^2-i\|u_1+u_2-iv\|^2\\&=2\|u_1+\frac{1}{2}v\|^2+2\|u_2+\frac{1}{2}v\|^2-\|u_1-u_2\|^2-2\|u_1-\frac{1}{2}v\|^2-2\|u_2-\frac{1}{2}v\|^2+\|u_1-u_2\|^2\\&+2i\|u_1+i\frac{1}{2}v\|^2+2i\|u_2+i\frac{1}{2}v\|^2-i\|u_1-u_2\|^2-2i|u_1-i\frac{1}{2}v\|^2-2i\|u_2-\frac{1}{2}v\|^2+i\|u_1-u_2\|^2\\&=8\langle u_1,\frac{v}{2}\rangle+8\langle u_2,\frac{1}{2}v\rangle=4\langle u_1,v\rangle+4\langle u_2,v\rangle.\end{aligned}$$

Thus $\langle,\rangle$ is pre-Hilbert inner product giving the original norm. □

iv) Let V be a Hilbert space, so as in (i) but complete as well. Let $C\subset V$ be a closed non-empty convex subset, meaning $v,w\in C\Rightarrow(v+w)/2\in C$. Show that there exists a unique $v\in C$ minimizing the norm, i.e. such that

$$\|v\|=\inf_{w\in C}\|w\|.$$

Hint: Use the parallelogram law to show that a norm minimizing sequence is Cauchy.

Solution. By definition of the infimum, there is a sequence $v_j \in C$ such that $\|v_j\| \to I = \inf_{w\in C} \|w\|$. By the parallelogram law

$$\|v_j - v_k\|^2 = 2\|v_j\|^2 + 2\|v_k\|^2 - 4\|\frac{1}{2}(v_j + v_k)\|^2 \le 2\|v_j\|^2 + 2\|v_k\|^2 - 4I \to 0$$

as $j, k \to \infty$, where the convexity of C has been used. Thus the sequence is Cauchy, so converges by the assumed completeness of the space. If v is the limit, the continuity of the norm implies that $|v\| = I$. If $w \in C$ is any point with $\|w\| = I$ then again by the parallelogram law

$$\|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2 - 4\|\frac{1}{2}(v + w)\|^2 \le 0$$

so $v = w$ and the point is unique. □

v) Let $u : H \to \mathbb{C}$ be a continuous linear functional on a Hilbert space, so $|u(\varphi)| \le C\|\varphi\| \ \forall \ \varphi \in H$. Show that $N = \{\varphi \in H; u(\varphi) = 0\}$ is closed and that if $v_0 \in H$ has $u(v_0) \neq 0$ then each $v \in H$ can be written uniquely in the form

$$v = cv_0 + w, \quad c \in \mathbb{C}, \quad w \in N.$$

Solution. That N is closed follows from the continuity of u, since $N = u^{-1}(0)$. If $a = u(v_0) \neq 0$ and $v \in H$ then $w = v - \frac{u(v)}{a} v_0 \in N$ so $v = cv_0 + w$ with $c = u(v)/u(v_0)$. □

vi) With u as in v), not the zero functional, show that there exists a unique $f \in H$ with $u(f) = 1$ and $\langle w, f\rangle = 0$ for all $w \in N$.
Hint: Apply iv) to $C = \{g \in V; u(g) = 1\}$.

Solution. If u is not the zero functional there exists $f \in H$ with $u(f) = 1$. Thus $C = \{g \in H; u(g) = 1\}$ is a closed set and it is convex since $u(\frac{1}{2}(g_1 + g_2)) = 1$ if $g_1, g_2 \in C$. Thus, by iv) there exists a unique $f \in C$ minimizing the norm. Now, by v) each $g \in C$ can be written uniquely $g = f + w$ with $u(w) = 0$. Since $f + sw \in C$ for all $s \in \mathbb{R}$ it follows that

$$\frac{d}{ds}\|f + sw\|^2\Big|_{s=0} = \langle f, w\rangle + \langle w, f\rangle = 0.$$

The same is true of iw so $\langle f, w\rangle = 0$ for all $w \in N$. Conversely if this holds for some point $g \in C$ then

$$\|g + w\|^2 = \|g\|^2 + \|w\|^2$$

is minimized on C at g; thus $g = f$ by the uniquenss of the minimizer. □

vii) Prove the Riesz Representation theorem, that every continuous linear functional on a Hilbert space is of the form

$$u_f : H \ni \varphi \mapsto \langle \varphi, f\rangle \text{ for a unique } f \in H.$$

Solution. If u is the zero functional this is clear, with $f = 0$. Thus we may assume that u is not identically zero, so v) applies, as does vi). Now for every $v \in H$, $v = u(v)f + w$ with $u(w) = 0$. Thus $\langle v, f\rangle = u(v)\|f\|^2$ and hence

$$u(v) = \langle v, g\rangle,\ g = f/\|f\|^2.$$

This g is unique since $f = g/\|g\|^2$ satisfies $\langle w, f\rangle = 0$ for all $w \in N$ and $u(f) = u(g)/\|g\|^2 = 1$, the solution of which is unique by vi). □

Problem 2. Density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$.

i) Recall in a few words why simple integrable functions are dense in $L^1(\mathbb{R}^n)$ with respect to the norm $\|f\|_{L^1} = \int_{\mathbb{R}^n} |f(x)|dx$.

Solution. We may divide and $f \in L^1(\mathbb{R}^n)$ into its real and imaginary parts and approximate them separately; this it suffices to consider real functions. We showed in class that any integrable function is the sum of a positive integrable function f_+ and a negative integrable function f_- (namely the positive and negative parts). For the positive function we showed that there is an increasing sequence of positive simple functions g_j converging to f_+ almost everywhere, so $\|f_+ - g_j\|_{L^1} \to 0$. Similarly for f_-. Thus f is the limit in L^1 of a sequence of simple integrable functions. □

ii) Show that simple functions $\sum_{j=1}^N c_j\chi(U_j)$ where the U_j are open and bounded are also dense in $L^1(\mathbb{R}^n)$.

Solution. We have seen that simple functions are dense, so it suffices to show that each element $c\chi(U)$ can be approximated in L^1 by simple functions with U_j open and bounded. Now, by integrability, $v(U) < \infty$ if $c \neq 0$, which we can assume. Then, by the definition of outer measure there is a sequence of measurable open sets $V_j \supset U$ with $v(V_j) \downarrow v(U)$. It follows that $\chi(V_j) \to \chi(U)$ in L^1. Now it is also the case that $v(V_j \cap \{|z| < R\}) \uparrow v(V_j)$ as $T \to \infty$ so simple functions with open bounded supporting sets are dense in L^1. □

iii) Show that if U is open and bounded then $F(y) = v(U \cap U_y)$, where $U_y = \{z \in \mathbb{R}^n; z = y + y', y' \in U\}$ is continuous in $y \in \mathbb{R}^n$ and that

$$v(U \cap U_y^{\complement}) + v(U^{\complement} \cap U_y) \to 0 \text{ as } y \to 0. \tag{2}$$

Solution. To prove continuity as $y \to 0$ consider the set

$$U_\delta = \{p \in U; |p - q| \le \delta \Longrightarrow q \in U\},\ \delta > 0.$$

Since the balls are closed, this is an open set and

$$U_\delta \subset U_y \cap U \subset U \text{ if } |y| \le \delta.$$

Furthermore, the U_δ increase as δ decreases and

$$\bigcup_{\delta>0} U_\delta = U.$$

If we set $V_n = U_{1/(n+1)} \setminus U_{1/n}$ the V_n are measurable and disjoint so

$$\sum_{n=1}^\infty v(V_n) = v(U) \Longrightarrow v(U_\delta) \to v(U) \text{ as } \delta \to 0.$$

The same argument applies to continuity at a general point $\bar{y}$. Simply take

$$U_{\bar{y},\delta} = \{p \in U; p + y \in U \text{ if } |y - \bar{y}| \le \delta\}.$$

These open sets increase as δ decreases with the union being $U \cap U_{\bar{y}}$.

Now to see (2), use the measurability of U and U_y to write

$$v(U) = v(U \cap U_y^{\complement}) + v(U \cap U_y) \to v(U)$$

as $|y| \to 0$, so $v(U \cap U_y^{\complement}) \to 0$ as $|y| \to 0$. Similarly for $v(U^{\complement} \cap U_y)$. □

iv) If U is open and bounded and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ show that
$$f(x) = \int_U \varphi(x-y)dy \in \mathcal{C}_c^\infty(\mathbb{R}^n).$$

Solution. Certainly f is bounded by $v(u)\sup|\varphi|$. Continuity of f follows from the fact that $\varphi(x_j - y)$ converges uniformly to $\varphi(x-y)$ for $y \in U$ as $x_j \to x$. Using difference quotients we can see that
$$D^\alpha f(x) = \int_U D^\alpha \varphi(x-y)dy$$
so all derivatives are also bounded and continuous. Clearly f vanishes outside the compact set $\operatorname{supp}(\varphi) + \bar{U}$. □

v) Show that if U is open and bounded then
$$\sup_{|y|\le\delta} \int |\chi_U(x) - \chi_U(x-y)|dx \to 0 \text{ as } \delta \downarrow 0.$$

Solution. Here, $\chi_U(x-y)$ is the characteristic function of the set $y + U = U_y$.The integral is of the simple function
$$\chi_{U\cap U_y^{\complement}} + \chi_{U^{\complement}\cap U_y}.$$
which by iii) converges to 0 in L^1. □

vi) If U is open and bounded and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\varphi \ge 0$, $\int \varphi = 1$ then
$$f_\delta \to \chi_U \text{ in } L^1(\mathbb{R}^n) \text{ as } \delta \downarrow 0$$
where
$$f_\delta(x) = \delta^{-n}\int \varphi\left(\frac{y}{\delta}\right)\chi_U(x-y)dy.$$
Hint: Write $\chi_U(x) = \delta^{-n}\int\varphi\left(\frac{y}{\delta}\right)\chi_U(x)dy$ and use v).

Solution. From the hint we see that
$$f_\delta(x) - \chi_U(x) = \delta^{-n}\int \varphi\left(\frac{y}{\delta}\right)\left(\chi_U(x-y) - \chi_U(x)\right)dy.$$
Using the positivity of φ,
$$\|f_\delta - \chi_U\|_{L^1} \le \delta^{-n}\int\varphi\left(\frac{y}{\delta}\right)\int|\chi_U(x-y) - \chi_U(x)|dxdy$$
By v) the inner integral converges uniformly to zero as $\delta \to 0$ so the whole thing is bounded by
$$\epsilon(\delta)\delta^{-n}\int\varphi\left(\frac{y}{\delta}\right)dy = \epsilon(\delta) \to 0$$
with δ. □

vii) Conclude that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.

Solution. We have already seen that simple functions which are sums of multiples of characteristic functions for open bounded sets are dense in L^1 and these in turn can be approximated in $L^1(\mathbb{R}^n)$ by elements of $\mathcal{C}_c^\infty(\mathbb{R}^n)$, so this space is dense in $L^1(\mathbb{R}^n)$. □

viii) Show that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for any $1 \le p < \infty$.

Solution. The argument follows closely that for L^1. First we may divide an L^p function into real and imaginary parts and approximate each separately. Then, for a real function, into its positive and negative parts, each of which is L^p. As before a non-negative L^p function is the a.e. limit of an increasing sequence of non-negative simple functions. By dominated convergence this shows that simple functions are dense in L^p. Thus it suffices to approximate χ_U be smooth functions of compact support and as above it suffices to take U open and bounded. Finally then we just need to redo step vi) for L^p. Again

$$\|f_\delta - \chi_U\|_{L^p} \le \delta^{-n} \int \varphi\left(\frac{y}{\delta}\right) \int |\chi_U(x-y) - \chi_U(x)|^p dx dy \to 0$$

as $\delta \to 0$. This proves the density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$. □

Problem 3. Schwartz representation theorem. Here we (well you) come to grips with the general structure of a tempered distribution.

i) Recall briefly the proof of the Sobolev embedding theorem and the corresponding estimate

$$\sup_{x\in\mathbb{R}^n} |\phi(x)| \le C\|\phi\|_{H^m},\ \frac{n}{2} < m \in \mathbb{R}. \tag{3}$$

Solution. If $u \in H^m(\mathbb{R}^n)$ then by definition

$$(1+|\xi|^2)^{m/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n).$$

Thus

$$\int |\hat{u}(\xi)| \le (\int (1+|\xi|^2)^{-m} d\xi)^{\frac{1}{2}} (\int (1+|\xi|^2)^m |\hat{u}(\xi)|^2 d\xi)^{\frac{1}{2}} < \infty$$

if $m > n/2$ so $\hat{u} \in L^1(\mathbb{R}^n)$. The inverse Fourier transform

$$u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi}\hat{u}(\xi)d\xi$$

is therefore a bounded function and (3) holds. It is continuous by an approximation argument. Namely we know that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R})$ and it follows from (3) applied to the difference that $u \in H^m(\mathbb{R}^n)$ is the uniform limit of a sequence of bounded continuous functions, hence is continuous.

Note that there is a slight falicy in this argument, since in reality u is only equal to a bounded continuous function almost everywhere. Can you see where this happens? □

ii) For $m = n+1$ write down a(n equivalent) norm on the right in a form that does not involve the Fourier transform.

Solution. A confusingly trivial question! By definition

$$\|u\|^2_{H^{n+1}} = \sum_{|\alpha|\le n+1} \|D^\alpha u\|^2_{L^2}.$$

□

iii) Show that for any $\alpha \in \mathbb{N}_0$

$$|D^\alpha \left((1+|x|^2)^N \phi\right)| \le C_{\alpha,N} \sum_{\beta\le\alpha} (1+|x|^2)^N |D^\beta \phi|. \tag{4}$$

Solution. Use Leibniz formula:

$$D^\alpha(f\phi) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} f D^\beta \phi. \tag{5}$$

The derivatives of $(1+|x|^2)^N$ are polynmomials of degree less than $2N$ so are each bounded by a constant multiple of $(1+|x|^2)^N$. Thus (4) follows from (5). □

iv) Deduce the general estimates

$$\sup_{\substack{|\alpha|\le N\\ x\in\mathbb{R}^n}} (1+|x|^2)^N |D^\alpha \phi(x)| \le C_N \|(1+|x|^2)^N \phi\|_{H^{N+n+1}}. \tag{6}$$

Solution. First we may reverse (4), by rearranging Leibniz formula, to see that

$$|(1+|x|^2)^N D^\alpha \phi| \le |D^\alpha \left((1+|x|^2)^N \phi\right)| + C_{\alpha,N} \sum_{\beta<\alpha} (1+|x|^2)^N |D^\beta \phi|.$$

Then proceeding inductively over α we can see that

$$|(1+|x|^2)^N D^\alpha \phi| \le \sum_{\beta\le\alpha} C_{N,\alpha} |D^\beta \left((1+|x|^2)^N \phi\right)|.$$

Now we may apply (3) to $D^\beta\left((1+|x|^2)^N\phi\right)$ on the right in (5) and arrive at (6) □

v) Conclude that for each tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ there is an integer N and a constant C such that

$$|u(\phi)| \le C \|(1+|x|^2)^N \phi\|_{H^{2N}} \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{7}$$

Solution. By assumption $u : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ is continuous, so $|u(\phi)| \le \|\phi\|_N$ for some N where the norm is bounded above by the left side of (6). Thus (7) follows; here I have increased N until $2N \ge N+n+1$. □

vi) Show that $v = (1+|x|^2)^{-N} u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

$$|v(\phi)| \le C \|(1+|D|^2)^N \phi\|_{L^2} \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{8}$$

Solution. From its definition and (7)

$$|v(\phi)| = |u((1+|x|^2)^{-N}\phi)| \le C\|\phi\|_{H^{2N}} = C\|(1+|D|^2)^N \phi\|_{L^2}.$$

□

vi) Recall (from class or just show it) that if v is a tempered distribution then there is a unique $w \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1+|D|^2)^N w = v$.

Solution. Using the Fourier transform, take $\hat{w} = (1+|\xi|^2)^{-N} \hat{v} \in \mathcal{S}'(\mathbb{R}^n)$. □

vii) Use the Riesz Representation Theorem to conclude that for each tempered distribution u there exists N and $w \in L^2(\mathbb{R}^n)$ such that

$$u = (1 + |D|^2)^N (1 + |x|^2)^N w. \tag{9}$$

Solution. If $u \in \mathcal{S}'(\mathbb{R}^n)$ consider v as in vi) and then w as in vi). It follows from (8) that

$$w(\hat{\phi}) = \hat{w}(\phi) = \hat{u}((1 + |\xi|^2)^{-N}\phi) \Longrightarrow |w(\hat{\phi})| \le C\|\hat{w}\|_{L^2} \le C'\|w\|_{L^2}.$$

Thus, w extends by continuity to be a continuous linear functional on $L^2(\mathbb{R}^n)$. By the Riesz Representation theorem, it is given by integration against and L^2 function, meaning it 'is' and L^2 function. Thus we finally conclude that

$$u = (1 + |x|^2)^N (1 + |D|^2)^N w, \ w \in L^2(\mathbb{R}^n). \tag{10}$$

This is probably what I meant to ask. Anyway, (9) follows by first taking the Fourier transform and getting a representation as in (10) for $\hat{u}$, then taking the inverse Fourier transform, which turns $(1+|x|^2)^N$ into $(1+|D|^2)^N$ and vice versa. □

viii) Use the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ (and the fact that it is an isomorphism on $L^2(\mathbb{R}^n)$) to show that any tempered distribution can be written in the form

$$u = (1 + |x|^2)^N (1 + |D|^2)^N w \text{ for some } N \text{ and some } w \in L^2(\mathbb{R}^n).$$

Solution. Right, backwards. □

ix) Show that any tempered distribution can be written in the form

$$u = (1 + |x|^2)^N (1 + |D|^2)^{N+n+1} \tilde{w} \text{ for some } N \text{ and some } \tilde{w} \in H^{2(n+1)}(\mathbb{R}^n).$$

Solution. Now replace w by $w = (1 + |D|^2)^{-n-1}\tilde{w}$ using the Fourier transform; clearly $\tilde{w} \in H^{2n+2}(\mathbb{R}^n)$. □

x) Conclude that any tempered distribution can be written in the form

$$u = (1 + |x|^2)^N (1 + |D|^2)^M U \text{ for some } N, M$$
$$\text{and a bounded continuous function } U$$

Solution. By the Sobolev embedding theorem, $\tilde{w}$ is just such a bounded continuous function. □

References

[1] George F. Simmons, *Introduction to topology and modern analysis*, Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1983, Reprint of the 1963 original. MR **84b:**54002

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