SIXTH ASSIGNMENT, DUE OCTOBER 23 IN CLASS
18.155 FALL 2001

RICHARD MELROSE

Problem 1. Hilbert space and the Riesz representation theorem. If you need help with this, it can be found in lots of places – for instance [1] has a nice treatment.

i) A pre-Hilbert space is a vector space V (over $\mathbb{C}$) with a 'positive definite sesquilinear inner product' i.e. a function

$$V \times V \ni (v, w) \mapsto \langle v, w\rangle \in \mathbb{C}$$

satisfying

- $\langle w, v\rangle = \overline{\langle v, w\rangle}$
- $\langle a_1 v_1 + a_2 v_2, w\rangle = a_1\langle v_1, w\rangle + a_2\langle v_2, w\rangle$
- $\langle v, v\rangle \ge 0$
- $\langle v, v\rangle = 0 \Rightarrow v = 0.$

Prove Schwarz' inequality, that

$$|\langle u, v\rangle| \le \langle u\rangle^{\frac{1}{2}}\langle v\rangle^{\frac{1}{2}} \ \forall\ u, v \in V.$$

Hint: Reduce to the case $\langle v, v\rangle = 1$ and then expand

$$\langle u - \langle u, v\rangle v, u - \langle u, v\rangle v\rangle \ge 0.$$

ii) Show that $\|v\| = \langle v, v\rangle^{1/2}$ is a norm and that it satisfies the parallelogram law:

$$\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 = 2\|v_1\|^2 + 2\|v_2\|^2 \ \forall\ v_1, v_2 \in V. \tag{1}$$

iii) Conversely, suppose that V is a linear space over $\mathbb{C}$ with a norm which satisfies (1). Show that

$$4\langle v, w\rangle = \|v + w\|^2 - \|v - w\|^2 + i\|v + iw\|^2 - i\|v - iw\|^2$$

defines a pre-Hilbert inner product which gives the original norm.

iv) Let V be a Hilbert space, so as in (i) but complete as well. Let $C \subset V$ be a closed non-empty convex subset, meaning $v, w \in C \Rightarrow (v+w)/2 \in C$. Show that there exists a unique $v \in C$ minimizing the norm, i.e. such that

$$\|v\| = \inf_{w\in C} \|w\|.$$

Hint: Use the parallelogram law to show that a norm minimizing sequence is Cauchy.

v) Let $u : H \to \mathbb{C}$ be a continuous linear functional on a Hilbert space, so $|u(\varphi)| \le C\|\varphi\| \ \forall\ \varphi \in H$. Show that $N = \{\varphi \in H; u(\varphi) = 0\}$ is closed and that if $v_0 \in H$ has $u(v_0) \ne 0$ then each $v \in H$ can be written uniquely in the form

$$v = cv_0 + w, \quad c \in \mathbb{C}, \quad w \in N.$$

vi) With u as in v), not the zero functional, show that there exists a unique $f \in H$ with $u(f) = 1$ and $\langle w, f\rangle = 0$ for all $w \in N$.
Hint: Apply iv) to $C = \{g \in V; u(g) = 1\}$.

vii) Prove the Riesz Representation theorem, that every continuous linear functional on a Hilbert space is of the form

$$u_f : H \ni \varphi \mapsto \langle \varphi, f\rangle \text{ for a unique } f \in H.$$

Problem 2. Density of $\mathcal{C}^\infty_c(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$.

i) Recall in a few words why simple integrable functions are dense in $L^1(\mathbb{R}^n)$ with respect to the norm $\|f\|_{L^1} = \int_{\mathbb{R}^n} |f(x)|dx$.

ii) Show that simple functions $\sum_{j=1}^N c_j\chi(U_j)$ where the U_j are open and bounded are also dense in $L^1(\mathbb{R}^n)$.

iii) Show that if U is open and bounded then $F(y) = v(U \cap U_y)$, where $U_y = \{z \in \mathbb{R}^n; z = y + y', y' \in U\}$ is continuous in $y \in \mathbb{R}^n$ and that

$$v(U \cap U_y^{\complement}) + v(U^{\complement} \cap U_y) \to 0 \text{ as } y \to 0.$$

Solution. To prove continuity as $y \to 0$ consider the set

$$U_\delta = \{p \in U; |p - q| \le \delta \Longrightarrow q \in U\},\ \delta > 0.$$

Since the balls are closed, this is an open set and

$$U_\delta \subset U_y \cap U \subset U \text{ if } |y| \le \delta.$$

Furthermore, the U_δ increase as δ decreases and

$$\bigcup_{\delta>0} U_\delta = U.$$

If we set $V_n = U_{1/(n+1)} \setminus U_{1/n}$ the V_n are measurable and disjoint so

$$\sum_{n=1}^{\infty} v(V_n) = v(U) \Longrightarrow v(U_\delta) \to v(U) \text{ as } \delta \to 0.$$

The same argument applies to continuity at a general point $\bar{y}$. Simply take

$$U_{\bar{y},\delta} = \{p \in U; p + y \in U \text{ if } |y - \bar{y}| \le \delta\}.$$

These open sets increas as δ decreases with the union being $U \cap U_{\bar{y}}$. □

iv) If U is open and bounded and $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ show that

$$f(x) = \int_U \varphi(x - y)dy \in \mathcal{C}^\infty_c(\mathbb{R}^n).$$

v) Show that if U is open and bounded then

$$\sup_{|y|\le\delta} \int |\chi_U(x) - \chi_U(x - y)|dx \to 0 \text{ as } \delta \downarrow 0.$$

vi) If U is open and bounded and $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\varphi \ge 0$, $\int \varphi = 1$ then

$$f_\delta \to \chi_U \text{ in } L^1(\mathbb{R}^n) \text{ as } \delta \downarrow 0$$

where

$$f_\delta(x) = \delta^{-n} \int \varphi\left(\frac{y}{\delta}\right) \chi_U(x - y)dy.$$

Hint: Write $\chi_U(x) = \delta^{-n} \int \varphi\left(\frac{y}{\delta}\right) \chi_U(x)$ and use v).

vii) Conclude that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.
viii) Show that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for any $1 \le p < \infty$.

Problem 3. Schwartz representation theorem. Here we (well you) come to grips with the general structure of a tempered distribution.

i) Recall briefly the proof of the Sobolev embedding theorem and the corresponding estimate
$$\sup_{x\in\mathbb{R}^n} |\phi(x)| \le C\|\phi\|_{H^m},\ \frac{n}{2} < m \in \mathbb{R}.$$
ii) For $m = n+1$ write down a(n equivalent) norm on the right in a form that does not involve the Fourier transform.
iii) Show that for any $\alpha \in \mathbb{N}_0$
$$|D^\alpha\left((1+|x|^2)^N\phi\right)| \le C_{\alpha,N}\sum_{\beta\le\alpha}(1+|x|^2)^N|D^\beta\phi|.$$
iv) Deduce the general estimates
$$\sup_{\substack{|\alpha|\le N\\ x\in\mathbb{R}^n}} (1+|x|^2)^N|D^\alpha\phi(x)| \le C_N\|(1+|x|^2)^N\phi\|_{H^{N+n+1}}.$$
v) Conclude that for each tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ there is an integer N and a constant C such that
$$|u(\phi)| \le C\|(1+|x|^2)^N\phi\|_{H^{2N}}\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$
vi) Show that $v = (1+|x|^2)^{-N}u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies
$$|v(\phi)| \le C\|(1+|D|^2)^N\phi\|_{L^2}\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$
vi) Recall (from class or just show it) that if v is a tempered distribution then there is a unique $w \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1+|D|^2)^N w = v$.
vii) Use the Riesz Representation Theorem to conclude that for each tempered distribution u there exists N and $w \in L^2(\mathbb{R}^n)$ such that

$$u = (1+|D|^2)^N(1+|x|^2)^N w. \tag{2}$$

viii) Use the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ (and the fact that it is an isomorphism on $L^2(\mathbb{R}^n)$) to show that any tempered distribution can be written in the form
$$u = (1+|x|^2)^N(1+|D|^2)^N w \text{ for some } N \text{ and some } w \in L^2(\mathbb{R}^n).$$
ix) Show that any tempered distribution can be written in the form
$$u = (1+|x|^2)^N(1+|D|^2)^{N+n+1}\tilde{w} \text{ for some } N \text{ and some } \tilde{w} \in H^{2(n+1)}(\mathbb{R}^n).$$
x) Conclude that any tempered distribution can be written in the form
$$u = (1+|x|^2)^N(1+|D|^2)^M U \text{ for some } N, M$$
and a bounded continuous function U

References

[1] George F. Simmons, *Introduction to topology and modern analysis*, Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1983, Reprint of the 1963 original. MR **84b:**54002

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu