THIRD ASSIGNMENT WITH SOLUTIONS (NOT PROOFREAD) WAS DUE SEPTEMBER 25 IN CLASS 18.155 FALL 2001

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Special note: Full marks may be achieved without doing the last three parts of the third problem.

Answers are a bit briefer this time, since I was busy making all the corrections!

These three problems are all about homogeneous distributions on the line, extending various things I did in class. We observed that

$$x_+^z = \begin{cases} \exp(z\log x) & x > 0 \\ 0 & x \le 0 \end{cases}$$

is a continuous function on $\mathbb{R}$ if $\operatorname{Re} z > 0$ and is differentiable if $\operatorname{Re} z > 1$ and then satisfies

$$\frac{d}{dx} x_+^z = z x_+^{z-1}.$$

We used this to define

$$x_+^z = \frac{1}{z+k}\frac{1}{z+k-1}\cdots\frac{1}{z+1}\frac{d^k}{dx^k}x_+^{z+k} \text{ if } z \in \mathbb{C}\setminus -\mathbb{N}. \tag{1}$$

Problem 1. [Hadamard regularization]

i) Show that (1) just means that for each $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$

$$x_+^z(\phi) = \frac{(-1)^k}{(z+k)\cdots(z+1)}\int_0^\infty \frac{d^k\phi}{dx^k}(x)x^{z+k}dx,\ \operatorname{Re} z > -k,\ z \notin -\mathbb{N}. \tag{2}$$

Solution. This is iteration of the definition of differentiation plus the fact that x_+^{z+k} is coninuous when $\operatorname{Re} z > -k$ so it is given by the integral. □

ii) Use integration by parts to show that

$$x_+^z(\phi) = \lim_{\epsilon\downarrow 0}\left[\int_\epsilon^\infty \phi(x)x^z dx - \sum_{j=1}^k C_j(\phi)\epsilon^{z+j}\right],\ \operatorname{Re} z > -k,\ z \notin -\mathbb{N} \tag{3}$$

for certain constants $C_j(\phi)$ which you should give explicitly. [This is called Hadamard regularization after Jacques Hadamard, feel free to look at his classic book [1].]

Solution. Integrating by parts

$$\int_\epsilon^\infty \frac{d^k\phi}{dx^k}(x)x^{z+k}dx$$
$$= (-1)^k(z+k)\cdots(z+1)\int_\epsilon^\infty \phi(x)x^z dx$$
$$-(z+k)\cdots(z+k-j+1)\sum_{j=0}^{k-1}(-1)^j\frac{d^{k-j}\phi}{dx^{k-j}}(\epsilon)\epsilon^{z+k-j}.$$

This gives (3) with

$$C_j(\phi) = \frac{(-1)^j}{(z+j)\cdots(z+1)}\frac{d^j\phi}{dx^j}(\epsilon).$$

Notice the we can drop the term with $j = k$ since, for $\operatorname{Re} z < -k$, the factor ϵ^{z+k} tends to 0 with ϵ. □

iii) Assuming that $-k+1 \geq \operatorname{Re} z > -k$, $z \neq -k+1$, show that there can only be one set of the constants with $j < k$ (for each choice of $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$) such that the limit in (3) exists.

Solution. This was originally incorrect. The point is simply that if there were two sets of constants then their differences d_j would have to satify

$$\lim_{\epsilon\downarrow 0}\sum_{j=1}^k d_j\epsilon^{z+j} = 0.$$

The largest term here corresponds to $j = 1$ so multiplying by ϵ^{-z-1}, which tends to zero with ϵ it follows that $d_1 = 0$, provided $k > 1$. One can continue this way to conclude that $d_j = 0$ for $j < k$. □

iv) Use ii), and maybe iii), to show that

$$\frac{d}{dx}x_+^z = zx_+^{z-1} \text{ in } \mathcal{C}^{-\infty}(\mathbb{R})\ \forall\ z \notin -\mathbb{N}_0 = \{0,1,\dots\}. \tag{4}$$

Proof. Use integration by parts in (3) to see that

$$\frac{dx_+^z}{dx}(\phi) = x_+^z(-\frac{d\phi}{dx})$$

is given by z times the same integral as for x_+^{z-1}. By iii) we do not need to compute the constants to see that (4) follows. □

v) Similarly show that $xx_+^z = x_+^{z+1}$ for all $z \notin -\mathbb{N}$.

Solution. Computing $x_+^z(x\phi)$ using (3) gives the same integral as for x_+^{z+1} for some constants, which we do not need to compute. □

vi) Show that $x_+^z = 0$ in $x < 0$ for all $z \notin -\mathbb{N}$. (Duh.)

Solution. The integral in (3) vanishes if ϕ has support in $x < 0$. □

Problem 2. [Null space of $x\frac{d}{dx} - z$]

i) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ then $\tilde{u}(\phi) = u(\tilde{\phi})$, where $\tilde{\phi}(x) = \phi(-x)\ \forall\ \phi \in \mathcal{C}^\infty_c(\mathbb{R})$, defines an element of $\mathcal{C}^{-\infty}(\mathbb{R})$. What is $\tilde{u}$ if $u \in \mathcal{C}^0(\mathbb{R})$? Compute $\widetilde{\delta_0}$.

Solution. For $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$, $\|\tilde{\phi}\|_m = \|\phi\|_m$ so continuity of the linear functional $\tilde{u}$ follows from that of u. Changing the variable of integration from x to $-x$ shows that $\tilde{u} = u(-x)$ if $u \in \mathcal{C}^0(\mathbb{R})$. Since $\tilde{\phi}(0) = \phi(0)$, $\widetilde{\delta_0} = \delta_0$. □

ii) Show that $\frac{d}{dx}\tilde{u} = -\widetilde{\frac{d}{dx}u}$.

Solution. This is true for test functions and hence for distributions. □

iii) Define $x^z_- = \widetilde{x^z_+}$ for $z \notin -\mathbb{N}$ and show that $\frac{d}{dx}x^z_- = -zx^{z-1}_-$ and $xx^z_- = -x^{z+1}_-$.

Solution. Follows from ii) and the fact that $x\tilde{\phi} = -\widetilde{x\phi}$ for test functions. □

iv) Suppose that $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ satisfies the distributional equation $(x\frac{d}{dx} - z)u = 0$ (meaning of course, $x\frac{du}{dx} = zu$ where z is a constant). Show that

$$u\big|_{x>0} = c_+x^z_-\big|_{x>0} \text{ and } u\big|_{x<0} = c_-x^z_-\big|_{x<0}$$

for some constants $c_\pm$. Deduce that $v = u - c_+x^z_+ - c_-x^z_-$ satisfies

$$(x\frac{d}{dx} - z)v = 0 \text{ and } \operatorname{supp}(v) \subset \{0\}. \tag{5}$$

Solution. Consider the distribution $w = x^{-z}u\big|_{x>0}$ on $x > 0$. It makes sense as x^{-z} is smooth on $x > 0$ and satisfies $\frac{dw}{dx} = -zw + x^{-z}\frac{du}{dx} = 0$. Thus, by the important result from class, $w = c_+$ is a contant in $x > 0$. The same argument applies in $x < 0$ with a possibly different constant. Then $v = u - c_+x^z_+ - c_-x^z_-$ has support in $\{0\}$ (since it vanishes on $x > 0$ and $x < 0$). Since $x^z_\pm$ satisfy the same equation as u so does v. □

v) Show that for each $k \in \mathbb{N}$, $(x\frac{d}{dx} + k + 1)\frac{d^k}{dx^k}\delta_0 = 0$.

Solution. Note the correction to the homogeneity. By definition

$$\begin{aligned}
x\frac{d}{dx}\frac{d^k}{dx^k}\delta_0(\phi) &= (-1)^{k+1}\frac{d^{k+1}}{dx^{k+1}}(x\phi)(0) \\
&= (-1)^{k+1}((k+1)\frac{d^k\phi}{dx^k} + x\frac{d^{k+1}}{dx^{k+1}}\phi)(0) = -(-1)^k(k+1)\frac{d^k\phi}{dx^k}(0) \\
&= -(k+1)\frac{d^k}{dx^k}(\delta_0)(\phi).
\end{aligned}$$

□

vi) Using the *fact* that any $v \in \mathcal{C}^{-\infty}(\mathbb{R})$ with $\operatorname{supp}(v) \subset \{0\}$ is a finite sum of constant multiples of the $\frac{d^k}{dx^k}\delta_0$, show that, for $z \notin -\mathbb{N}$, the only solution of (5) is $v = 0$.

Solution. As shown in class the support condition in (5) implies that $v = \sum_{j=0}^{N} c_j \frac{d^j}{dx^j}\delta_0$ where we may assume that $c_N \neq 0$ unless $v = 0$ as a distribution. Applying $xd/dx - z$ to both sides and using the preceeding result we conclude that

$$\sum_{j=0}^{N}(j+1-z)c_j\frac{d^j}{dx^j}\delta_0 = 0$$

as a distribution. Pairing this with a test function which has its first $N-1$ derivatives zero at 0, but not its Nth (e.g. $x^N\psi$ where ψ is our little bump function around 0) we see that $(j+1-z)c_N = 0$. Since $j+1 \neq z$ it follows that $c_N = 0$ which is a contradiction, so $v = 0$. □

vii) Conclude that for $z \notin -\mathbb{N}$

(6) $$\left\{u \in \mathcal{C}^{-\infty}(\mathbb{R}); (x\frac{d}{dx} - z)u = 0\right\}$$

is a two-dimensional vector space.

Solution. We know that $c_1 x_+^z + c_2 x_-^z$ lies in the space for any constants c_1 and c_2. Since one of these vanishes in $x < 0$ and the other in $x > 0$ it cannot be zero (as a distribution) unless both c_1 and c_2 vanish. Thus the space is at least two dimensional. The uniqueness discussion above shows that it is precisely two dimensional since every solution is of this form. □

Problem 3. [Negative integral order] To do the same thing for negative integral order we need to work a little differently. Fix $k \in \mathbb{N}$.

i) We define *weak convergence* of distributions by saying $u_n \to u$ in $\mathcal{C}_c^\infty(X)$, where $u_n,\ u \in \mathcal{C}^{-\infty}(X)$, $X \subset \mathbb{R}^n$ being open, if $u_n(\phi) \to u(\phi)$ for each $\phi \in \mathcal{C}_c^\infty(X)$. Show that $u_n \to u$ implies that $\frac{\partial u_n}{\partial x_j} \to \frac{\partial u}{\partial x_j}$ for each $j = 1, \dots, n$ and $fu_n \to fu$ if $f \in \mathcal{C}^\infty(X)$.

Solution. $u_n \to u$ weakly means tha t$u_n(\phi) \to u(\phi)$ for each $\phi \in \mathcal{C}_c^\infty(X)$. In particular $u_n(-\frac{\partial\phi}{\partial x_j}) \to u(-\frac{\partial\phi}{\partial x_j})$ which means $\frac{\partial u_n}{\partial x_j} \to \frac{\partial u}{\partial x_j}$ weakly. □

ii) Show that $(z+k)x_+^z$ is weakly continuous as $z \to -k$ in the sense that for any sequence $z_n \to -k$, $z_n \notin -\mathbb{N}$, $(z_n+k)x_+^{z_n} \to v_k$ where

$$v_k = \frac{1}{-1}\cdots\frac{1}{-k+1}\frac{d^{k+1}}{dx^{k+1}}x_+,\ x_+ = x_+^1.$$

Solution. If $u_n = x_+^{z_n}$ then from (2)

$$(z+k)u_n(\phi) = \frac{(-1)^k}{(z_n+k-1)\cdots(z+1)}\int_0^\infty \frac{d^k\phi}{dx^k}(x)x^{z_n+k}dx \to$$
$$\frac{(-1)^k}{(-1)\cdots(-k+1)}\int_0^\infty \frac{d^k\phi}{dx^k}(x)dx$$
$$= -\frac{1}{(-1)\cdots(-k+1)}\frac{d^{k-1}}{dx^{k-1}}\delta_0(\phi) = v_k(\phi).$$

Note that the definition of v_k was wrong; also I am cheating just a little here, since I should really use a representation of x_+^z which is valid for $\operatorname{Re} z < -k$ as well. Increase k to $k+1$ and do the computation! □

iii) Compute v_k, including the constant factor.

Solution. I just did. □

iv) Do the same thing for $(z+k)x_-^z$ as $z \to -k$.

Proof. Since $x_-^z = \widetilde{x_+^z}$ we can observe that $u_n \to u$ weakly imples $\widetilde{u_n} \to \tilde{u}$ weakly, so $(z_n + k)x_-^{z_n} \to \widetilde{v_k} = (-1)^{k-1} v_k$. □

v) Show that there is a linear combination $(k+z)(x_+^z + c(k)x_-^z)$ such that as $z \to -k$ the limit is zero.

Solution. Take $c(k) = (-1)^k$ and the limit will be zero. □

vi) If you get this far, show that in fact $x_+^z + c(k)x_-^z$ also has a weak limit, u_k, as $z \to -k$. [This may be the hardest part.]

Proof. What, you expect me to work for nothing? Okay, observe that $(z+k)x_+^z(\phi)$ restricted to $z \in (-k, -k + \frac{1}{2})$ is actually a differentiable function with derivative which is continuous up to $z = -k$. [Of course it is really a holomorphic function of z as a complex variable near $-k$; if you do not understand this then you should take the time to learn a little complex analysis some time, it is worth the effort and essential for a working mathematician.] To see the differentiability, just differentiate (2) with respect to z (after multiplying by $(z+k)$. Now, the same differentiability is true for $(k+z)(x_+^z + (-1)^k x_-^z)(\phi)$ and its limit is zero as $z \to -k$. Thus

$$w_k(\phi) = \lim_{z \downarrow -k} (x_+^z + (-1)^k x_-^z)(\phi)$$

exists for each $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. Furthermore it is a distribution (from the continuity estimates on the derivative.) □

vii) Show that this limit distribution satisfies $(x\frac{d}{dx} + k)u_k = 0$.

Solution. Since $x_+^z + (-1)^k x_-^z$ is homogeneous of degree z, w_k is homogeneous of degree $-k$ by weak convergence:

$$w_k((-\frac{d}{dx}x + k)\phi) = \lim_{z \to -k} ((x_+^z + (-1)^k x_-^z))((-\frac{d}{dx}x - z)\phi) = 0$$

It cannot be zero since it restricts to x^{-k} on $x > 0$. □

viii) Conclude that (6) does in fact hold for $z \in -\mathbb{N}$ as well. [There are still some things to prove to get this.]

Solution. We know that the null space of $x\frac{d}{dx} + k$ (the space of solutions) is at least two dimensional, since it contains $c_1 w_k + c_1 \frac{d^{k-1}}{dx^{k-1}}\delta_0$. This combination cannot vanish as a distribution unless both coefficients vanish. So it remains to show that there are no other solutions. The argument in 2 vii) shows that any element of the null space must restrict to $c_+ x^{-k}$ in $x > 0$ and similarly in $x < 0$ for a different constant. Subtracting the correct multple of w_k we can arrange that $u = 0$ in $x < 0$. Now, remember that the limit of $(z+k)x_+^z$ is v_k so, again by Taylor's theorem

$$\gamma_k(\phi) = \lim_{z \to -k} x_+^z - \frac{1}{z+k} v_k(\phi)$$

exists as a weak limit. This distribution restricts to x^{-k} in $x > 0$ but is *not* homogeneous. In fact it satisfies

$$(x\frac{d}{dx} + k)\gamma_k = c_k \frac{d^{k-1}}{dx^{k-1}}\delta_0,\ \ c_k \neq 0.$$

From this it follows that there is no distribution which vanishes in $x < 0$, is equal to x^{-k} in $x > 0$ and is homogeneous of degree $-k$. Indeed, if w were such a distribution then $w - \gamma_k$ would have support in the origin, hence be a sum of derivatives of δ_0 must satisfy

$$(x\frac{d}{dx} + k)(w - \gamma_k) = -c_k \frac{d^{k-1}}{dx^{k-1}}\delta_0,\ \ c_k \neq 0.$$

By inspection there is no such distribution. Thus the null space of solutions of $x\frac{d}{dx} + k$ is also two dimensional when $k \in \mathbb{N}$. □

References

[1] J. Hadamard, *Le problème de Cauchy et les èquatons aux dérivées partielles linéaires hyperboliques*, Hermann, Paris, 1932.

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