FIRST ASSIGNMENT WITH SOLUTIONS, DUE SEPTEMBER 11 IN CLASS 18.155 FALL 2001

RICHARD MELROSE

In the solutions of these problems I am looking for precise statements and clear, succinct proofs. Some of these problem may involve things you do not know – of course, as with the anonymous quiz, I am simply trying to check your level of knowledge so that I can adjust the course as necessary. No weight, in terms of final grade, will be given to this assignment but please do it anyway and not anonymously this time.

Problem 1. [Taylor's theorem]. Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued function which is k times continuously differentiable. Prove that there is a polynomial p and a continuous function v such that

$$u(x) = p(x) + v(x) \text{ where } \lim_{|x|\downarrow 0} \frac{|v(x)|}{|x|^k} = 0.$$

Solution. The standard argument is to evaluate

$$R_k(x) = \frac{1}{(k-1)!}\int_0^1 (1-t)^{k-1}\left(\frac{d^k}{dt^k}u(tx)\right)dt$$

by integration by parts k times:

$$\begin{aligned} R_k(x) = \frac{1}{(k-1)!}\int_0^1 (1-t)^{k-1}\frac{d}{dt}\left(\frac{d^{k-1}}{dt^{k-1}}u(tx)\right)dt \\ = \left[\frac{(1-t)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dt^{k-1}}u(tx)\right]_{t=0}^{t=1} + R_{k-1}(x) \\ = u(x) - p_{k-1}(x) \end{aligned}$$

where p_{k-1} is clearly a polynomial (of degree $k-1$ in fact). Expanding the derivatives in $R_k(x)$ it may be written as

$$R_k(x) = \sum_{|\alpha|=k} x^\alpha R_\alpha(x)$$

where the R_α are continuous, so we may write $R_\alpha(x) = R_\alpha(0) + v_\alpha$ where v_α is continuous and vanishes at 0 and get the result with p being the Taylor polynomial of degree k of u and

$$v(x) = \sum_{|\alpha|=k} x^\alpha v_\alpha(x).$$

□

Problem 2. Let $\mathcal{C}(\mathbb{B}^n)$ be the space of continuous functions on the (closed) unit ball, $\mathbb{B}^n = \{x \in \mathbb{R}^n; |x| \le 1\}$. Let $\mathcal{C}_0(\mathbb{B}^n) \subset \mathcal{C}(\mathbb{B}^n)$ be the subspace of functions which vanish at each point of the boundary and let $\mathcal{C}(\mathbb{S}^{n-1})$ be the space of continuous functions on the unit sphere. Show that inclusion and restriction to the boundary gives a short exact sequence

$$\mathcal{C}_0(\mathbb{B}^n) \hookrightarrow \mathcal{C}(\mathbb{B}^n) \longrightarrow \mathcal{C}(\mathbb{S}^{n-1}) \tag{1}$$

(meaning the first map is injective, the second is surjective and the image of the first is the null space of the second.)

Solution. To prove the surjectivity of the second map in (1) we need to extend any given continuous function on the sphere to a continuous function on the ball. Choose $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ (continuous is enough) with $\phi(x) = 1$ in $|x| < \epsilon$ for some $\epsilon > 0$ and $\phi(x) = 0$ for $|x| > \frac{1}{2}$. (Such a function was constructed in class). Then if u is a continuous function on the sphere set

$$\tilde{u}(x) = (1 - \phi(x)u(\frac{x}{|x|}).$$

This is continuous near 0 since it vanishes in $|x| < \epsilon$. For $|x| > 0$ it is continuous as the product of two continuous functions. Clearly $\tilde{u}(x) = u(x)$ if $|x| = 1$. This proves the surjectivity of the second map.

The null space of the second map is precisely the set of functions which vanish at each point on the sphere, i.e. $\mathcal{C}_0(\mathbb{B}^n)$, so the sequence is exact. □

Problem 3. [Measures] A measure on the ball is a continuous linear functional $\mu : \mathcal{C}(\mathbb{B}^n) \longrightarrow \mathbb{R}$ where continuity is with respect to the supremum norm, i.e. there must be a constant C such that

$$|\mu(f)| \le C \sup_{x \in \mathbb{R}^n} |f(x)| \; \forall \; f \in \mathcal{C}(\mathbb{B}^n). \tag{2}$$

Let $M(\mathbb{B}^n)$ be the linear space of such measures. The space $M(\mathbb{S}^{n-1})$ of measures on the sphere is defined similarly. Describe an injective map

$$M(\mathbb{S}^{n-1}) \longrightarrow M(\mathbb{B}^n).$$

Can you define another space so that this can be extended to a short exact sequence?

Solution. If $u \in M(\mathbb{S}^{n-1})$ we may define $u' \in M(\mathbb{B}^n)$ using the second map in (1):

$$u'(\phi) = u(\phi\big|_{\mathbb{S}^{n-1}}), \; \phi \in \mathcal{C}(\mathbb{B}^n).$$

We do need to check that this defines a measure, but this follows from the continuity of u since

$$|u'(\phi)| = |u(\phi\big|_{\mathbb{S}^{n-1}})| \le C \sup_{\mathbb{S}^{n-1}} |\phi| \le C \sup_{\mathbb{B}^{n-1}} |\phi|.$$

Note that $u' = 0$ in $M(\mathbb{B}^n)$ certainly implies that $u = 0$, so the map is injective.

We may define the space $\tilde{M}(\mathbb{B}^n)$ as the space of continuous linear maps

$$u : \mathcal{C}_0(\mathbb{B}^n) \longrightarrow \mathbb{C}$$

where continuity means the same thing, (2), but only for $f \in \mathcal{C}_0(\mathbb{B}^n)$. We get a map $M(\mathbb{B}^n) \longrightarrow \tilde{M}(\mathbb{B}^n)$ by restricting $\mu \in M(\mathbb{B}^n)$ to $\mathcal{C}_0(\mathbb{B}^n)$. Then

$$M(\mathbb{S}^{n-1}) \longrightarrow M(\mathbb{B}^{n-1}) \longrightarrow \tilde{M}(\mathbb{B}^n)$$

is exact. I suppose I did not really ask you to show this, but it is not too hard. The null space of the second map consists of all the measure which vanish on $\mathcal{C}_0(\mathbb{B}^n)$. This means that if u is in the null space we can define an element $v \in M(\mathbb{S}^{n-1})$ by $v(\phi) = u(\phi')$ where for $\phi \in \mathcal{C}(\mathbb{S}^{n-1})$, $\phi' \in \mathcal{C}(\mathbb{B}^n)$ satisfies $\phi'\big|_{\mathbb{S}^{n-1}} = \phi$. The result is well-defined since $u(\phi')$ does not depend on which ϕ' we choos. Continuity of v

follows from the fact that we can choose ϕ' so $\sup_{\mathbb{B}^n} |\phi'| \le \sup_{\mathbb{S}^{n-1}} |\phi|$. This shows exactness in the middle and surjectivity of the second map follows similarly. The second sequence is in fact the dual of the first sequence. □

Problem 4. Show that the Riemann integral defines a measure

$$\mathcal{C}(\mathbb{B}^n) \ni f \longmapsto \int_{\mathbb{B}^n} f(x)dx. \tag{3}$$

Solution. This is the standard estimate on the Riemann integral,

$$|\int_{\mathbb{B}^n} f dx| \le \mathrm{Vol}(\mathbb{B}^n) \sup \mathbb{B}^n |f(x)|.$$

□

Problem 5. If $g \in \mathcal{C}(\mathbb{B}^n)$ and $\mu \in M(\mathbb{B}^n)$ show that $g\mu \in M(\mathbb{B}^n)$ where $(g\mu)(f) = \mu(fg)$ for all $f \in \mathcal{C}(\mathbb{B}^n)$. Describe all the measures with the property that

$$x_j\mu = 0 \text{ in } M(\mathbb{B}^n) \text{ for } j = 1, \dots, n.$$

Solution. The first part is just the product estimate:

$$\sup |fg| \le \sup |f| \dot{\sup} |g|$$

which shows that $g\mu$ is a measure since

$$|g\mu(f)| = |\mu(gf)| \le C \sup |fg| \le (c \sup |g|) \sup |f|.$$

The answer to the last part is that $\mu = \delta_0$ is the Dirac measure

$$\delta_0(f) = f(0).$$

To prove this, one should first show that if $f \in \mathcal{C}(\mathbb{B}^n)$ vanishes at 0 then $f = \lim_j f_j$ uniformly (i.e. in terms of the supremum norm) where each $f_j \in \mathcal{C}(\mathbb{B}^n)$ is of the form $f_j = x_1 g_{1j} + \cdots + x_n g_{nj}$ with the $g_{ij} \in \mathcal{C}(\mathbb{B}^n)$. Then if follows that $\mu(f) = 0$ for such f. The rest of the proof is the same as a question on the second assignment – which is actually easier than this one! □

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu