ELEVENTH AND LAST ASSIGNMENT, DUE DECEMBER 11 IN CLASS
18.155 FALL 2001

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This problem set is optional, but you can use it to improve your final marks.

Problem 1. [Separable Hilbert spaces]

i) (Gramm-Schmidt Lemma). Let $\{v_i\}_{i\in\mathbb{N}}$ be a sequence in a Hilbert space H. Let $V_j \subset H$ be the span of the first j elements and set $N_j = \dim V_j$. Show that there is an orthonormal sequence $e_1, \dots, e_j$ (finite if N_j is bounded above) such that V_j is the span of the first N_j elements. Hint: Proceed by induction over N such that the result is true for all j with $N_j < N$. So, consider what happens for a value of j with $N_j = N_{j-1} + 1$ and add element $e_{N_j} \in V_j$ which is orthogonal to all the previous e_k's.
ii) A Hilbert space is separable if it has a countable dense subset (sometimes people say Hilbert space when they mean separable Hilbert space). Show that every separable Hilbert space has a complete orthonormal sequence, that is a sequence $\{e_j\}$ such that $\langle u, e_j\rangle = 0$ for all j implies $u = 0$.
iii) Let $\{e_j\}$ an orthonormal sequence in a Hilbert space, show that for any $a_j \in \mathbb{C}$,

$$\|\sum_{j=1}^{N} a_j e_j\|^2 = \sum_{j=1}^{N} |a_j|^2.$$

iv) (Bessel's inequality) Show that if e_j is an orthormal sequence in a Hilbert space and $u \in H$ then

$$\|\sum_{j=1}^{N}\langle u, e_j\rangle e_j\|^2 \le \|u\|^2$$

and conclude (assuming the sequence of e_j's to be infinite) that the series

$$\sum_{j=1}^{\infty}\langle u, e_j\rangle e_j$$

converges in H.
v) Show that if e_j is a complete orthonormal basis in a separable Hilbert space then, for each $u \in H$,

$$u = \sum_{j=1}^{\infty}\langle u, e_j\rangle e_j.$$

Problem 2. [Compactness] Let's agree that a compact set in a metric space is one for which every open cover has a finite subcover. You may use the compactness of closed bounded sets in a finite dimensional vector space.

i) Show that a compact subset of a Hilbert space is closed and bounded.
ii) If e_j is a complete orthonormal subspace of a separable Hilbert space and K is compact show that given $\epsilon > 0$ there exists N such that

$$\sum_{j \geq N} |\langle u, e_j \rangle|^2 \leq \epsilon \ \forall \ u \in K. \tag{1}$$

iii) Conversely show that any closed bounded set in a separable Hilbert space for which (1) holds for some orthonormal basis is indeed compact.
iv) Show directly that any sequence in a compact set in a Hilbert space has a convergent subsequence.
v) Show that a subspace of H which has a precompact unit ball must be finite dimensional.
vi) Use the existence of a complete orthonormal basis to show that any bounded sequence $\{u_j\}$, $\|u_j\| \leq C$, has a weakly convergent subsequence, meaning that $\langle v, u_j \rangle$ converges in $\mathbb{C}$ along the subsequence for each $v \in H$. Show that the subsequnce can be chosen so that $\langle e_k, u_j \rangle$ converges for each k, where e_k is the complete orthonormal sequence.

Problem 3. [Spectral theorem, compact case] Recall that a bounded operator A on a Hilbert space H is compact if $A\{\|u\| \leq 1\}$ is precompact (has compact closure). Throughout this problem A will be a compact operator on a separable Hilbert space, H.

i) Show that if $0 \neq \lambda \in \mathbb{C}$ then

$$E_\lambda = \{u \in H; Au = \lambda u\}.$$

is finite dimensional.
ii) If A is self-adjoint show that all eigenvalues (meaning $E_\lambda \neq \{0\}$) are real and that different eigenspaces are orthogonal.
iii) Show that $\alpha_A = \sup\{|\langle Au, u \rangle|^2\}; \|u\| = 1\}$ is attained. Hint: Choose a sequence such that $|\langle Au_j, u_j \rangle|^2$ tends to the supremum, pass to a weakly convergent sequence as discussed above and then using the compactness to a furhter subsequence such that Au_j converges.
iv) If v is such a maximum point and $f \perp v$ show that $\langle Av, f \rangle + \langle Af, v \rangle = 0$.
v) If A is also self-adjoint and u is a maximum point as in iii) deduce that $Au = \lambda u$ for some $\lambda \in \mathbb{R}$ and that $\lambda = \pm\alpha$.
vi) Still assuming A to be self-adjoint, deduce that there is a finite-dimensional subspace $M \subset H$, the sum of eigenspaces with eigenvalues $\pm\alpha$, containing all the maximum points.
vii) Continuing vi) show that A restricts to a self-adjoint bounded operator on the Hilbert space $M^\perp$ and that the supremum in iii) for this new operator is smaller.
viii) Deduce that for any compact self-adjoint operator on a separable Hilbert space there is a complete orthonormal basis of eigenvectors. Hint: Be careful about the null space – it could be big.

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