Problem 1. (Poisson summation formula) As in class, let \( L \subset \mathbb{R}^n \) be an integral lattice of the form
\[
L = \left\{ v = \sum_{j=1}^{n} k_j v_j, \ k_j \in \mathbb{Z} \right\}
\]
where the \( v_j \) form a basis of \( \mathbb{R}^n \) and using the dual basis \( w_j \) (so \( w_j \cdot v_i = \delta_{ij} \) is 0 or 1 as \( i \neq j \) or \( i = j \)) set
\[
L^\circ = \left\{ w = 2\pi \sum_{j=1}^{n} k_j w_j, \ k_j \in \mathbb{Z} \right\}.
\]
Recall that we defined
\[
(1) \quad C^\infty(\mathbb{T}_L) = \{ u \in C^\infty(\mathbb{R}^n); u(z + v) = u(z) \ \forall \ z \in \mathbb{R}^n, \ v \in L \}.
\]
i) Show that summation over shifts by lattice points:
\[
(2) \quad A_L : \mathcal{S}(\mathbb{R}^n) \ni f \mapsto A_L f(z) = \sum_{v \in L} f(z - v) \in C^\infty(\mathbb{T}_L).
\]
defines a map into smooth periodic functions.
ii) Show that there exists \( f \in C^\infty_c(\mathbb{R}^n) \) such that \( A_L f \equiv 1 \) is the constant function on \( \mathbb{R}^n \).
iii) Show that the map (2) is surjective. Hint: Well obviously enough use the \( f \) in part ii) and show that if \( u \) is periodic then \( A_L(uf) = u \).
iv) Show that the infinite sum
\[
(3) \quad F = \sum_{v \in L} \delta(\cdot - v) \in \mathcal{S}'(\mathbb{R}^n)
\]
does indeed define a tempered distribution and that \( F \) is \( L \)-periodic and satisfies \( \exp(iw \cdot z)F(z) = F(z) \) for each \( w \in L^\circ \) with equality in \( \mathcal{S}'(\mathbb{R}^n) \).
v) Deduce that \( \hat{F} \), the Fourier transform of \( F \), is \( L^\circ \)-periodic, conclude that it is of the form
\[
(4) \quad \hat{F}(\xi) = c \sum_{w \in L^\circ} \delta(\xi - w)
\]
vi) Compute the constant \( c \).
vii) Show that \( A_L(f) = F \ast f \).
viii) Using this, or otherwise, show that \( A_L(f) = 0 \) in \( C^\infty(\mathbb{T}_L) \) if and only if \( \hat{f} = 0 \) on \( L^\circ \).
Problem 2. For a measurable set $\Omega \subset \mathbb{R}^n$, with non-zero measure, set $H = L^2(\Omega)$ and let $\mathcal{B} = \mathcal{B}(H)$ be the algebra of bounded linear operators on the Hilbert space $H$ with the norm on $\mathcal{B}$ being

\[
\|B\|_{\mathcal{B}} = \sup\{\|Bf\|_H; f \in H, \|f\|_H = 1\}.
\]

i) Show that $\mathcal{B}$ is complete with respect to this norm. Hint (probably not necessary!) For a Cauchy sequence $\{B_n\}$ observe that $B_n f$ is Cauchy for each $f \in H$.

ii) If $V \subset H$ is a finite-dimensional subspace and $W \subset H$ is a closed subspace with a finite-dimensional complement (that is $W + U = H$ for some finite-dimensional subspace $U$) show that there is a closed subspace $Y \subset W$ with finite-dimensional complement (in $H$) such that $V \perp Y$, that is $\langle v, y \rangle = 0$ for all $v \in V$ and $y \in Y$.

iii) If $A \in \mathcal{B}$ has finite rank (meaning $AH$ is a finite-dimensional vector space) show that there is a finite-dimensional space $V \subset H$ such that $AV \subset V$ and $AV^\perp = \{0\}$ where

\[
V^\perp = \{f \in H; \langle f, v \rangle = 0 \ \forall \ v \in V\}.
\]

Hint: Set $R = AH$, a finite dimensional subspace by hypothesis. Let $N$ be the null space of $A$, show that $N^\perp$ is finite dimensional. Try $V = R + N^\perp$.

iv) If $A \in \mathcal{B}$ has finite rank, show that $(\text{Id} - zA)^{-1}$ exists for all but a finite set of $\lambda \in \mathbb{C}$ (just quote some matrix theory). What might it mean to say in this case that $(\text{Id} - zA)^{-1}$ is meromorphic in $z$? (No marks for this second part).

v) Recall that $\mathcal{K} \subset \mathcal{B}$ is the algebra of compact operators, defined as the closure of the space of finite rank operators. Show that $\mathcal{K}$ is an ideal in $\mathcal{B}$.

vi) If $A \in \mathcal{K}$ show that

\[
\text{Id} + A = (\text{Id} + B)(\text{Id} + A')
\]

where $B \in \mathcal{K}$, $(\text{Id} + B)^{-1}$ exists and $A'$ has finite rank. Hint: Use the invertibility of $\text{Id} + B$ when $\|B\|_{\mathcal{B}} < 1$ proved in class.

vii) Conclude that if $A \in \mathcal{K}$ then

\[
\{f \in H; (\text{Id} + A)f = 0\} \text{ and } ((\text{Id} + A)H)^\perp
\]

are finite dimensional.