

**TENTH ASSIGNMENT, DUE DECEMBER 4 IN CLASS
18.155 FALL 2001**

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Problem 1. (Poisson summation formula) As in class, let $L \subset \mathbb{R}^n$ be an integral lattice of the form

$$L = \left\{ v = \sum_{j=1}^n k_j v_j, k_j \in \mathbb{Z} \right\}$$

where the v_j form a basis of \mathbb{R}^n and using the dual basis w_j (so $w_j \cdot v_i = \delta_{ij}$ is 0 or 1 as $i \neq j$ or $i = j$) set

$$L^\circ = \left\{ w = 2\pi \sum_{j=1}^n k_j w_j, k_j \in \mathbb{Z} \right\}.$$

Recall that we defined

$$(1) \quad \mathcal{C}^\infty(\mathbb{T}_L) = \{u \in \mathcal{C}^\infty(\mathbb{R}^n); u(z+v) = u(z) \forall z \in \mathbb{R}^n, v \in L\}.$$

i) Show that summation over shifts by lattice points:

$$(2) \quad A_L : \mathcal{S}(\mathbb{R}^n) \ni f \longmapsto A_L f(z) = \sum_{v \in L} f(z-v) \in \mathcal{C}^\infty(\mathbb{T}_L).$$

defines a map into smooth periodic functions.

ii) Show that there exists $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $A_L f \equiv 1$ is the constant function on \mathbb{R}^n .

iii) Show that the map (2) is surjective. Hint: Well obviously enough use the f in part ii) and show that if u is periodic then $A_L(uf) = u$.

iv) Show that the infinite sum

$$(3) \quad F = \sum_{v \in L} \delta(\cdot - v) \in \mathcal{S}'(\mathbb{R}^n)$$

does indeed define a tempered distribution and that F is L -periodic and satisfies $\exp(iw \cdot z)F(z) = F(z)$ for each $w \in L^\circ$ with equality in $\mathcal{S}'(\mathbb{R}^n)$.

v) Deduce that \hat{F} , the Fourier transform of F , is L° periodic, conclude that it is of the form

$$(4) \quad \hat{F}(\xi) = c \sum_{w \in L^\circ} \delta(\xi - w)$$

vi) Compute the constant c .

vii) Show that $A_L(f) = F \star f$.

viii) Using this, or otherwise, show that $A_L(f) = 0$ in $\mathcal{C}^\infty(\mathbb{T}_L)$ if and only if $\hat{f} = 0$ on L° .

Problem 2. For a measurable set $\Omega \subset \mathbb{R}^n$, with non-zero measure, set $H = L^2(\Omega)$ and let $\mathcal{B} = \mathcal{B}(H)$ be the algebra of bounded linear operators on the Hilbert space H with the norm on \mathcal{B} being

$$(5) \quad \|B\|_{\mathcal{B}} = \sup\{\|Bf\|_H; f \in H, \|f\|_H = 1\}.$$

- i) Show that \mathcal{B} is complete with respect to this norm. Hint (probably not necessary!) For a Cauchy sequence $\{B_n\}$ observe that $B_n f$ is Cauchy for each $f \in H$.
- ii) If $V \subset H$ is a finite-dimensional subspace and $W \subset H$ is a closed subspace with a finite-dimensional complement (that is $W + U = H$ for some finite-dimensional subspace U) show that there is a closed subspace $Y \subset W$ with finite-dimensional complement (in H) such that $V \perp Y$, that is $\langle v, y \rangle = 0$ for all $v \in V$ and $y \in Y$.
- iii) If $A \in \mathcal{B}$ has finite rank (meaning AH is a finite-dimensional vector space) show that there is a finite-dimensional space $V \subset H$ such that $AV \subset V$ and $AV^\perp = \{0\}$ where

$$V^\perp = \{f \in H; \langle f, v \rangle = 0 \forall v \in V\}.$$

Hint: Set $R = AH$, a finite dimensional subspace by hypothesis. Let N be the null space of A , show that N^\perp is finite dimensional. Try $V = R + N^\perp$.

- iv) If $A \in \mathcal{B}$ has finite rank, show that $(\text{Id} - zA)^{-1}$ exists for all but a finite set of $\lambda \in \mathbb{C}$ (just quote some matrix theory). What might it mean to say in this case that $(\text{Id} - zA)^{-1}$ is meromorphic in z ? (No marks for this second part).
- v) Recall that $\mathcal{K} \subset \mathcal{B}$ is the algebra of compact operators, defined as the closure of the space of finite rank operators. Show that \mathcal{K} is an ideal in \mathcal{B} .
- vi) If $A \in \mathcal{K}$ show that

$$\text{Id} + A = (\text{Id} + B)(\text{Id} + A')$$

where $B \in \mathcal{K}$, $(\text{Id} + B)^{-1}$ exists and A' has finite rank. Hint: Use the invertibility of $\text{Id} + B$ when $\|B\|_{\mathcal{B}} < 1$ proved in class.

- vii) Conclude that if $A \in \mathcal{K}$ then

$$\{f \in H; (\text{Id} + A)f = 0\} \text{ and } ((\text{Id} + A)H)^\perp \text{ are finite dimensional.}$$