In the solutions of these problems I am looking for precise statements and clear, succinct proofs. Some of these problems may involve things you do not know – of course, as with the anonymous quiz, I am simply trying to check your level of knowledge so that I can adjust the course as necessary. No weight, in terms of final grade, will be given to this assignment but please do it anyway and not anonymously this time.

Problem 1

[Taylor’s theorem]. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function which is $k$ times continuously differentiable. Prove that there is a polynomial $p$ and a continuous function $v$ such that

$$u(x) = p(x) + v(x) \text{ where } \lim_{|x| \to 0} \frac{|v(x)|}{|x|^k} = 0.$$

Problem 2

Let $C(B^n)$ be the space of continuous functions on the (closed) unit ball, $B^n = \{x \in \mathbb{R}^n; |x| \leq 1\}$. Let $C_0(B^n) \subset C(B^n)$ be the subspace of functions which vanish at each point of the boundary and let $C(S^{n-1})$ be the space of continuous functions on the unit sphere. Show that inclusion and restriction to the boundary gives a short exact sequence

$$C_0(B^n) \to C(B^n) \to C(S^{n-1})$$

(meaning the first map is injective, the second is surjective and the image of the first is the null space of the second.)

Problem 3

[Measures] A measure on the ball is a continuous linear functional $\mu : C(B^n) \to \mathbb{R}$ where continuity is with respect to the supremum norm, i.e. there must be a constant $C$ such that

$$|\mu(f)| \leq C \sup_{x \in \mathbb{R}^n} |f(x)| \forall f \in C(B^n).$$

Let $M(B^n)$ be the linear space of such measures. The space $M(S^{n-1})$ of measures on the sphere is defined similarly. Describe an injective map

$$M(S^{n-1}) \to M(B^n).$$

Can you define another space so that this can be extended to a short exact sequence?
Problem 4

Show that the Riemann integral defines a measure

\[ \mathcal{C}(\mathbb{B}^n) \ni f \mapsto \int_{\mathbb{B}^n} f(x) \, dx. \]

Problem 5

If \( g \in \mathcal{C}(\mathbb{B}^n) \) and \( \mu \in \mathcal{M}(\mathbb{B}^n) \) show that \( g\mu \in \mathcal{M}(\mathbb{B}^n) \) where \( (g\mu)(f) = \mu(fg) \) for all \( f \in \mathcal{C}(\mathbb{B}^n) \). Describe all the measures with the property that

\[ x_j \mu = 0 \quad \text{in} \quad \mathcal{M}(\mathbb{B}^n) \quad \text{for} \quad j = 1, \ldots, n. \]

Department of Mathematics, Massachusetts Institute of Technology

E-mail address: rbm@math.mit.edu