PREPARATION FOR TEST 1 FOR 18.102
TEST ON FEBRUARY 27, 2020

The test will consist of some of the questions below. Hints on request.

No papers, books, electronic divices or such may be consulted during the test.

You may use the results we have shown in class up to this point including Monotonicity, Fatou, Lebesgue Dominated Convergence, the result that an absolutely summable sequence in $\mathcal{L}^1(\mathbb{R})$ converges almost everywhere and the completeness of $L^1(\mathbb{R})$.

Question 1. Show that the rational numbers form a set of measure in $\mathbb{R}$.

Question 2. Show that the function

$$u(x) = \begin{cases} 0 & x \le 0 \\ \min(x^{-\frac{1}{2}}, x^{-2}) & x > 0 \end{cases} \tag{1}$$

is in $\mathcal{L}^1(\mathbb{R})$.

Question 3. Show that the product of a bounded continuous function on $\mathbb{R}$ and an element of $\mathcal{L}^1(\mathbb{R})$ is in $\mathcal{L}^1(\mathbb{R})$.

Question 4. Show that if $t \in \mathbb{R}$ and $f \in \mathcal{L}^1(\mathbb{R})$ then

$$f_t(x) = f(x-t) \tag{2}$$

is an element of $\mathcal{L}^1(\mathbb{R})$. Prove that $f \in \mathcal{L}^1(\mathbb{R})$ is *continuous-in-the-mean* in the sense that given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|t| < \delta \Longrightarrow \int |f_t - f| < \epsilon. \tag{3}$$

Question 5. Suppose that $B : L^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$ is a bounded linear operator. Show that if $B(\phi) = 0$ for all $\phi \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ then $B = 0$ as an operator.

Question 6. Find the real numbers s such that $(1 + |x|)^{s/2} \in L^1(\mathbb{R})$ and justify your conclusion.

Question 7. Suppose that $[f_j] \in L^1(\mathbb{R})$ is a Cauchy sequence, show that f_j has a subsequence which converges almost everywhere.

Question 8. We know that the characteristic function of a finite interval, $\chi_{[a,b)}$, $a < b$ real, is integrable. A (real-valued) step function is a finite sum of real multiples of such characteristic functions. If a function $f : [a, b] \longrightarrow \mathbb{R}$, on a finite interval, is *Riemann integrable* then it is easy to see from the definition (you do not need to check this) that given $\epsilon > 0$ there are two step functions U and L such that

$$L(x) \le f(x) \le U(x) \text{ on } [a, b] \text{ and } \int U - \int L < \epsilon \tag{4}$$

where we are using the Lebesgue integral. Deduce that a Riemann integrable function on an interval $[a, b]$, extended as zero outside its domain of definition, is Lebesgue integrable.

Question 9. Suppose $g \in \mathcal{L}^1(\mathbb{R})$ is non-negative and vanishes outside some bounded interval. Show that if $t \in (0,1)$ then

$$g_t(x) = \begin{cases} 0 & \text{if } g(x) = 0 \\ g(x)^t & \text{if } g(x) > 0 \end{cases} \tag{5}$$

is an element of $\mathcal{L}^1(\mathbb{R})$ and that $\lim_{t\downarrow 0} g_t(x) = \chi_V(x)$ is the characteristic function of some set and that the limit is in $\mathcal{L}^1(\mathbb{R})$.

Question 10. Suppose that $f_n \in \mathcal{L}^1(\mathbb{R})$ is absolutely summable, $\sum\limits_n \int |f_n| < \infty$. Show that the convergence

$$f(x) = \sum_n f_n(x) \ a.e.$$

is dominated by an L^1 function.

Question 11. Prove that the function

$$u(x) = \begin{cases} x^{-1} & x > 0 \\ 0 & x \le 0 \end{cases}$$

is *not* in $\mathcal{L}^1(\mathbb{R})$.

Question 12. We say that a subset $A \subset \mathbb{R}$ has finite Lebesgue measure if its characteristic function

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{6}$$

is in $\mathcal{L}^1(\mathbb{R})$ and then its measure is $\mu(A) = \int \chi_A$. Show that the union of two such sets, A and B, has finite Lebesgue measure and that

$$\mu(A \cup B) \le \mu(A) + \mu(B). \tag{7}$$