

**PROBLEM SET 8 FOR 18.102, SPRING 2018
DUE FRIDAY 24 APRIL.**

RICHARD MELROSE

Problem 8.1

Using the fact shown in class that the $(\sqrt{2\pi})^{-1}e^{ikx}$, for $k \in \mathbb{Z}$, form an orthonormal basis of $L^2(0, \pi)$, go through the proof that for appropriate constants $c_n > 0$, $n \in \mathbb{N}$, the functions $c_n \sin(nx)$ form an orthonormal basis of $L^2(0, \pi)$.

Hint: This is done in the notes.

Problem 8.2

Define $H_0^2([0, \pi]) \subset L^2(0, \pi)$ as consisting of those functions for which the (un-normalized) Fourier-Bessel coefficients (for the basis introduced above) satisfy

$$(1) \quad \sum_{k \in \mathbb{N}_0} |k^2 c_k|^2 < \infty \text{ where } c_k = \int_0^\pi \sin kx f(x) dx.$$

This is a *Sobolev space*. I'm not sure what the H stands for (Hilbert maybe) but the superscript '2' stands for two derivatives (in L^2) and the subscript 0 means vanishing at the boundary – see below! The same space is sometimes denoted $L^{2,2}$.

Show that if $u \in H_0^2([0, \pi])$ and u_N is the sum of the first N terms in the Fourier-Bessel series for u (which is in $L^2(0, \pi)$) then

$$(2) \quad u_N \rightarrow u, \quad \frac{du_N}{dx} \rightarrow F_1, \quad \frac{d^2 u_N}{dx^2} \rightarrow F_2$$

where in the first two cases we have convergence in supremum norm and in the third, convergence in $L^2(0, \pi)$. Deduce that $u \in C^0[0, \pi]$, $u(0) = u(\pi) = 0$ and $F_1 \in C^0[0, \pi]$ whereas $F_2 \in L^2(0, \pi)$.

Problem 8.3

Let $\mathcal{C}_0^2([0, \pi])$ be the space of twice continuously differentiable functions on $[0, \pi]$ (one-sided derivatives at the ends) which vanish at 0 and π – this is the space considered in lecture for the Dirichlet problem. Show that $H_0^2([0, \pi])$ is a Hilbert space with respect to the norm

$$(3) \quad \|u\|^2 = \sum_k (1 + k^4) |c_k|^2$$

and that $\mathcal{C}_0^2([0, \pi]) \subset H_0^2([0, \pi])$ is a dense subspace.

Hint: Try not to belabour the proof of completeness since you have done so many – but really do it nevertheless! If $\phi \in \mathcal{C}_0^2([0, \pi])$ compute the integrals in (1) above and integrate by parts to show the rest of (1). Think about $\sin kx$ (maybe write down a related orthonormal basis of $H_0^2([0, \pi])$) to prove density.

Problem 8.4

With F_1 and F_2 as in (2) for $u \in H_0^2([0, \pi])$ show that

$$(4) \quad \int_{0,\pi} u\phi' = - \int_{0,\pi} F_1\phi, \quad \int_{0,\pi} u\phi'' = \int_{0,\pi} F_2\phi, \quad \forall \phi \in \mathcal{C}_0^2([0, \pi])$$

and show that if $u \in \mathcal{C}_0^2([0, \pi]) \subset H_0^2([0, \pi])$ then $F_1 = u'$, $F_2 = u''$.

Hint: Try Cauchy-Schwartz inequality on the sum of the first N terms in the Fourier-Bessel series using the inequalities from (1).

Problem 8.5

Show that if $V \in \mathcal{C}^0[0, \pi]$ then the linear map

$$(5) \quad Q_V : H_0^2([0, \pi]) \ni u \mapsto -F_2 + Vu \in L^2(0, \pi)$$

is bounded and reduces to

$$(6) \quad u \mapsto -u'' + Vu \text{ on } \mathcal{C}_0^2([0, \pi]).$$

Problem 8.6-extra

Show that if V is real-valued then for $t \in i\mathbb{R}$ the map $Q_V + t\text{Id}$ is surjective and has a bounded inverse.

Problem 8.7 – extra

Define the Sobolev space $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$ of functions on the circle as the space of 2π -periodic functions on the line such that the Fourier coefficients satisfy

$$(7) \quad \sum_{k \in \mathbb{Z}} k^4 |c_k|^2 < \infty.$$

show it is a Hilbert space with an appropriate choice of the norm and that the 2π -periodic functions which are twice continuously differentiable form a dense subspace.