PROBLEM SET 2 FOR 18.102, SPRING 2020
BRIEF SOLUTIONS.

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1. Problem 2.1

Show that if $K \in \mathcal{C}([0,1]^2)$ is a continuous function of two variables, then the integral operator

$$Au(x) = \int_0^1 K(x,y)u(y)dy \tag{1}$$

(given by a Riemann integral) is a bounded operator, i.e. a continous linear map, from $\mathcal{C}([0,1])$ to itself with respect to the supremum norm.

Solution: A continuous function on a compact set, such as $[0,1]^2$, is uniformly continuous, so given ϵ there exists $\delta > 0$ such that

$$|x - x'| + |y - y'| < \delta \Longrightarrow |K(x,y) - K(x',y')| < \epsilon. \tag{2}$$

If $u \in \mathcal{C}([0,1])$ is fixed then the integrand in (1) is continuous for each fixed $x \in [0,1]$ so $Au : [0,1] \longrightarrow \mathbb{C}$ is well-defined as a Riemann integral. Moreover

$$|Au(x) - Au(x')| = |\int_0^1 (K(x,y) - K(x',y)u(y)dy| \le \sup_y |K(x,y) - K(x',y)| \sup |u|$$

by standard properties of the Riemann integral. Using (2) it follows that

$$|x - x'| < \delta \Longrightarrow |Au(x) - Au(x')| \le \sup |u| \epsilon$$

so Au is continous on $[0,1]$ and (1) defines a map

$$A : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1]). \tag{3}$$

The linearity of this map follows from the linearity of the Riemann integral and

$$|u(x)| \le \sup |K| \sup |u| \ \forall\ x \in [0,1] \tag{4}$$

shows that it is bounded, i.e. continuous.

2. Problem 2.2

(1) Show that the 'Dirac delta function at $y \in [0,1]$' is well-defined as a continuous linear map

$$\delta_y : \mathcal{C}([0,1]) \ni u \longmapsto u(y) \in \mathbb{C} \tag{1}$$

with respect to the supremum norm on $\mathcal{C}([0,1])$.

(2) Show that δ_y is *not* continuous with respect to the L^1 norm $\int_0^1 |u|$.

Solution

(1) The map (1) is clearly linear since

$$\delta_y(c_1u_1 + c_2u_2) = (c_1u_1 + c_2u_2)(y) = c_1\delta_y(u_1) + c_2\delta_u(u_2) \tag{2}$$

and it is bounded

$$|\delta_u(u)| \le \sup |u|$$

so continuous.

(2) It suffices to show that there is a sequence u_n in $\mathcal{C}([0,1])$ such that $\delta_y(u_n) = 1$ but $\|u_n\|_{L^1} \to 0$ since then a bound

$$|\delta_y(u)| \le C\|u\|_{L^1}$$

is impossible. Such a sequence is given by the 'triangle functions'

$$u_n(x) = \begin{cases} 0 & x \le y - 1/n \\ 1 - n|y-x| & y - 1/n \le x \le y + 1/n \\ 0 & x \ge y + 1/n \end{cases}$$

restricted to $[0,1]$. Indeed u_n is continuous at each point and

$$u_n(y) = 1, \int_0^1 u_n(y) \le 1/n. \tag{3}$$

3. Problem 2.3

A subset $E \subset \mathbb{R}$ is said to be *of measure zero* if there exists an absolutely summable sequence $f_n \in \mathcal{C}_c(\mathbb{R})$ (so $\sum\limits_n \int |f_n| < \infty$) such that

$$E \subset \{x \in \mathbb{R}; \sum_n |f_n(x)| = +\infty\}. \tag{1}$$

Show that if E is of measure zero and $\epsilon > 0$ is given then there exists $f_n \in \mathcal{C}_c(\mathbb{R})$ satisfying (1) and in addition

$$\sum_n \int |f_n| < \epsilon. \tag{2}$$

Solution: Take such a series f_n with $\sum\limits_n \int |f_n(x)| = C$ and replace it by $\frac{\epsilon}{C+1} f_n$ or choose N so large that

$$\sum_{n \le N} \int |f_n(x)| > C - \epsilon$$

and consider the new series $u_n = f_{n+N}$ which has

$$\sum_n \int |u_n(x)| < \epsilon \tag{3}$$

and for which $\sum\limits_n |u_n(x)|C$ diverges wherever $\sum\limits_n |f_n(x)|$ diverges, so in particular on E.

4. Problem 2.4

Using the previous problem (or otherwise ...) show that a countable union of sets of measure zero is a set of measure zero.

Solution: Let E_j be the countable collection of sets of measure zero. Choose a summable series $f_{j,n}$ for each j which satisfies

$$\sum_n \int |f_{j,n}| < 2^{-j}, \ \sum_n |f_{j,n}(x)| = \infty \text{ for } x \in E_j. \tag{1}$$

Now, rearrange the countably many terms $f_{j,n}$ into a sequence $g_k \in \mathcal{C}_c(\mathbb{R})$ – using for instance a bijection from $\mathbb{N}^2$ to $\mathbb{N}$ applied to the indices. Then, standard rearrangement properties of absolutely summable series (look at Rudin if you need to, we will use this next week) show that

$$\begin{gathered}\sum_k \int |g_k| = \sum_j \sum_n \int |f_{j,n}| < \sum_j 2^{-j} = 2, \\ \sum_k |g_k(x)| \geq \sum_n |f_{j,n}(x)| = \infty \ \forall \ x \in E_j, \ \forall \ j.\end{gathered} \tag{2}$$

Thus $E = \sum\limits_j E_j$ has measure zero.

Problem 2.5

Suppose $E \subset \mathbb{R}$ has the following (well-known) property:-

$$\begin{gathered}\forall \ \epsilon > 0 \ \exists \text{ a countable collection of intervals } (a_i, b_i) \text{ s.t.} \\ \sum_i (b_i - a_i) < \epsilon, \ E \subset \bigcup_i (a_i, b_i).\end{gathered} \tag{3}$$

Show that E is a set of measure zero in the sense used in lectures and above.

Solution: for $\epsilon_n = 1/n^2$, we have a countable collection of intervals $(a_i^{(n)}, b_i^{(n)})$ as in the question. Now define $f_i^{(n)}$ be 1 on $[a_i^{(n)}, b_i^{(n)}]$ and 0 outside $[a_i^{(n)} - \frac{b_i^{(n)} - a_i^{(n)}}{2}, b_i^{(n)} + \frac{b_i^{(n)} - a_i^{(n)}}{2}]$, and define the values elsewhere using linear segment. Then it's easy to verify $\int |f_i^{(n)}| = 2(b_i^{(n)} - a_i^{(n)})$, so $\sum_n \sum_i \int |f_i^{(n)}| < +\infty$, but for any $x \in E$, $\sum_n \sum_i \int |f_i^{(n)}(x)| = +\infty$ as $\sum_i \int |f_i^{(n)}(x)| \geq 1$ by definition. So E is of measure zero.

5. Problem 2.6 – Extra

Let's generalize the theorem about $\mathcal{B}(V, W)$ given last week to bilinear maps – this may seem hard but just take it step by step!

(1) Check that if U and V are normed spaces then $U \times V$ (the linear space of all pairs (u, v) where $u \in U$ and $v \in V$) is a normed space where addition and scalar multiplication is 'componentwise' and the norm is the sum

$$\|(u, v)\|_{U \times V} = \|u\|_U + \|v\|_V. \tag{1}$$

(2) Show that $U \times V$ is a Banach space if both U and V are Banach spaces.

(3) Consider three normed spaces U, V and W. Let

$$B : U \times V \longrightarrow W \tag{2}$$

be a *bilinear* map. This means that

$$B(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 B(u_1, v) + \lambda_2 B(u_2, v),$$
$$B(u, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 B(u, v_1) + \lambda_2 B(u, v_2)$$

for all u, u_1, $u_2 \in U$, v, v_1, $v_2 \in V$ and λ_1, $\lambda_2 \in \mathbb{C}$. Show that B is continuous if and only if it satisfies

$$\|B(u,v)\|_W \le C\|u\|_U \|v\|_V \; \forall \; u \in U, \; v \in V. \tag{3}$$

(4) Let $\mathcal{M}(U, V; W)$ be the space of all such continuous bilinear maps. Show that this is a linear space and that

$$\|B\| = \sup_{\|u\|=1, \|v\|=1} \|B(u,v)\|_W \tag{4}$$

is a norm.

(5) Show that $\mathcal{M}(U, V; W)$ is a Banach space if W is a Banach space.

Solution: Third last part only and brief. An estimate (3) implies continuity, since if $u_n \to u$ and $v_n \to v$ then

(5)
$$\|B(u_n, v_n) - B(u, v)\|_W \le \|B(u_n, v_n) - B(u_n, v)\|_W + \|B(u_n, v) - B(u, v)\|_W$$
$$\le C(\|u_n\|\|v_n - v\| + \|u_n - u\|\|u\|) \to 0.$$

Conversely, if B is continuous then $B^{-1}(\{\|w\| < 1\}) \ni 0$ is open, so

$$\|u\| + \|v\| < \epsilon \Longrightarrow \|B(u,v)\| \le 1$$

for some $\epsilon > 0$. If u and v are non-zero then

$$\|\epsilon/4(\frac{u}{\|u\|}, \frac{v}{\|v\|}) < \epsilon \Longrightarrow \|B(u,v)\| \le \frac{4}{\epsilon}\|u\|\|v\|$$

using the bilinearity. If either vanishe then $B(u, v)$ vanishes so (3) is equivalent to continuity.

Everything else is very similar to the linear case.

6. Problem 2.7 – Extra

Consider the space $\mathcal{C}_\mathrm{c}(\mathbb{R}^n)$ of continuous functions $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ which vanish outside a compact set, i.e. in $|x| > R$ for some R (depending on u). Check (quickly) that this is a linear space.

Show that if $y \in \mathbb{R}^{n-1}$ and $u \in \mathcal{C}_\mathrm{c}(\mathbb{R}^n)$ then

$$U_y : \mathbb{R} \ni t \longmapsto u(y, t) \in \mathbb{C} \tag{1}$$

defines an element $U_y \in \mathcal{C}_\mathrm{c}(\mathbb{R})$. Fix an overall 'rectangle' $[-R, R]^n$ and only consider functions $\mathcal{C}_{\mathrm{c},R}(\mathbb{R})$ vanishing outside this rectangle. With this restriction on supports show for each R that $\mathbb{R}^{n-1} \ni y \longmapsto U_y$ is a continuous map into $\mathcal{C}_{\mathrm{c},R}(\mathbb{R})$ *with respect to the supremum norm* which vanishes for $|y| > R$, i.e. has compact support. Conclude that 'integration in the last variable' gives a continuous linear map (with respect to supremum norms)

$$\mathcal{C}_{\mathrm{c},R}(\mathbb{R}^n) \ni u \longrightarrow v \in \mathcal{C}_{\mathrm{c},R}(\mathbb{R}^{n-1}), \; v(y) = \int U_y. \tag{2}$$

By iterating this statement show that the iterated Riemann integral is well defined

$$\int : \mathcal{C}_{c,R}(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{3}$$

and that $\int |u|$ is a norm which is independent of R – so defined on the whole of $\mathcal{C}_c(\mathbb{R}^n)$.

Solution: $y \mapsto U_y$ is continuous as $[-R, R]^n$ is compact so u is uniformly continuous then one easily gets the bound. The iterated Riemann integral is a norm: non-negative, absolute homogeneity, triangle inequality follows immediately, if $u \neq 0$, then $|u| > 0$ in an open neighborhood of some points, hence the integral is positive. The independence on R is because u vanishes outside the rectangle.