PROBLEM SET 1 FOR 18.102, SPRING 2020
BRIEF SOLUTIONS

RICHARD MELROSE

1. Problem 1.1

Write out a proof for each p with $1 \le p < \infty$ that

$$l^p = \left\{ a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty,\ a_j = a(j) \right\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold. Note that the only 'tricky' part is the triangle inequality for this all you really need in the way of 'hard estimates' is to show that (for all N)

$$\left(\sum_{j=1}^{N} |a_j|^p \right)^{\frac{1}{p}} \text{ is a norm on } \mathbb{C}^N. \tag{1}$$

I'm expecting that you will look up and give a brief proof of (1).

Solution: We know that the functions from any set with values in a linear space form a linear space – under addition of values (don't feel bad if you wrote this out, it is a good thing to do once). So, to see that l^p is a linear space it suffices to see that it is closed under addition and scalar multiplication. For scalar multiples this is clear:-

$$|ta_i| = |t||a_i| \text{ so } \|ta\|_p = |t|\|a\|_p \tag{2}$$

which is part of what is needed for the proof that $\|\cdot\|_p$ is a norm anyway. The fact that $a,\, b \in l^p$ imples $a + b \in l^p$ follows once we show the triangle inequality or we can be a little cruder and observe that

$$\begin{gathered} |a_i + b_i|^p \le (2\max(|a|_i, |b_i|))^p = 2^p \max(|a|_i^p, |b_i|^p) \le 2^p(|a_i|^p + |b_i|^p) \text{ so} \\ \|a + b\|_p^p = \sum_j |a_i + b_i|^p \le 2^p(\|a\|_p^p + \|b\|_p^p) \end{gathered} \tag{3}$$

where we use the fact that t^p is an increasing function of $t \ge 0$.

Now, to see that l^p is a normed space we need to check that $\|a\|_p$ is indeed a norm. It is non-negative and $\|a\|_p = 0$ implies $a_i = 0$ for all i which is to say $a = 0$.

So, only the triangle inequality remains. For $p = 1$ this is a direct consequence of the usual triangle inequality:

$$\|a+b\|_1 = \sum_i |a_i + b_i| \le \sum_i (|a_i| + |b_i|) = \|a\|_1 + \|b\|_1. \tag{4}$$

For $1 < p < \infty$ it is known as Minkowski's inequality. This in turn is deduced from Hölder's inequality – which follows from Young's inequality! The latter says if $1/p + 1/q = 1$, so $q = p/(p-1)$, then

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q} \ \forall\ \alpha, \beta \ge 0. \tag{5}$$

To check it, observe that as a function of $\alpha = x$,

$$f(x) = \frac{x^p}{p} - x\beta + \frac{\beta^q}{q} \tag{6}$$

if non-negative at $x = 0$ and clearly positive when $x >> 0$, since x^p grows faster than $x\beta$. Moreover, it is differentiable and the derivative only vanishes at $x^{p-1} = \beta$, where it must have a global minimum in $x > 0$. At this point $f(x) = 0$ so Young's inequality follows. Now, applying this with $\alpha = |a_i|/\|a\|_p$ and $\beta = |b_i|/\|b\|_q$ (assuming both are non-zero) and summing over i gives Hölder's inequality

$$\begin{aligned} |\sum_i a_i b_i|/\|a\|_p\|b\|_q \le \sum_i |a_i||b_i|/\|a\|_p\|b\|_q \le \sum_i \left(\frac{|a_i|^p}{\|a\|_p^p p} + \frac{|b_i|^q}{\|b\|_q^q q} \right) = 1 \\ \Longrightarrow |\sum_i a_i b_i| \le \|a\|_p \|b\|_q. \end{aligned} \tag{7}$$

Of course, if either $\|a\|_p = 0$ or $\|b\|_q = 0$ this inequality holds anyway.

Now, from this Minkowski's inequality follows. Namely from the ordinary triangle inequality and then Minkowski's inequality (with q power in the first factor)

$$\begin{aligned} \sum_i |a_i + b_i|^p &= \sum_i |a_i + b_i|^{(p-1)} |a_i + b_i| \\ &\le \sum_i |a_i + b_i|^{(p-1)} |a_i| + \sum_i |a_i + b_i|^{(p-1)} |b_i| \\ &\le \left(\sum_i |a_i + b_i|^p \right)^{1/q} (\|a\|_p + \|b\|_q) \end{aligned} \tag{8}$$

gives after division by the first factor on the right

$$\|a+b\|_p \le \|a\|_p + \|b\|_p. \tag{9}$$

Thus, l^p is indeed a normed space.

I did not necessarily expect you to go through the proof of Young-Hölder-Minkowksi, but I think you should do so at some point since I will not do it in class.

2. Problem 1.2

Prove directly that each l^p as defined in Problem 1.1 is a Banach space.

Remarks (for those who need orientation): This means showing that each Cauchy sequence converges; you need to mentally untangle the fact that we are talking about a sequence of sequences. The problem here is to find the limit of a given

Cauchy sequence. The usual approach is show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements (which are sequences) at the Nth term gives a Cauchy sequence with respect to the norm coming from (1) on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it really is the limit. You need to put all this together.

So, suppose we are given a Cauchy sequence $a^{(n)}$ in l^p. Thus, each element is a sequence $\{a_j^{(n)}\}_{j=1}^{\infty}$ in l^p. Now, each entry in a vector in l^p is bounded by the norm, so $|a_j^{(n)}| \le \|a^{(n)}\|_p$ and the same applies to the difference, so for each fixed j,

$$|a_j^{(n)} - a_j^{(m)}| \le \|a^{(n)} - a^{(m)}\| \tag{1}$$

and it follows that $a_j^{(n)}$ is a Cauchy sequence in $\mathbb{C}$, hence convergent. Let $a_j = \lim_{n\to\infty} a_j^{(n)}$ be the limit. Our putative limit is a, the sequence $\{a_i\}_{i=1}^{\infty}$. The boundedness of the norm of a Cauchy sequence shows that for some constant A such that $\|a^{(n)}\|_p \le A$ for all n and N,

$$\sum_{i=1}^{N} |a_i^{(n)}|^p \le A^p. \tag{2}$$

We can pass to the limit here as $n \to \infty$ since there are only finitely many terms. Thus

$$\sum_{i=1}^{N} |a_i|^p \le A^p \ \forall\ N \Longrightarrow \|a\|_p \le A. \tag{3}$$

Thus, $a \in l^p$ as we hoped. Similarly, we can pass to the limit as $m \to \infty$ in the finite inequality which follows from the Cauchy conditions

$$\left(\sum_{i=1}^{N} |a_i^{(n)} - a_i^{(m)}|^p\right)^{\frac{1}{p}} < \epsilon/2 \tag{4}$$

to see that for each N

$$\left(\sum_{i=1}^{N} |a_i^{(n)} - a_i|^p\right)^{\frac{1}{p}} \le \epsilon/2 \tag{5}$$

and hence

$$\|a^{(n)} - a\| < \epsilon \ \forall\ n > M. \tag{6}$$

Thus indeed, $a^{(n)} \to a$ in l^p as we were trying to show.

Notice that the trick is to 'back off' to finite sums to avoid any issues of interchanging limits.

3. Problem 1.3

Consider the 'unit sphere' in l^p. This is the set of vectors of length 1 :

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) Show that S is closed.
(2) Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e.g. by checking in Rudin).

(3) Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a 1 in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!

Solution:
The norm on any normed space is continuous since

$$|\|x\| - \|y\|| \leq \|x-y\| = d(x,y) \tag{1}$$

is the 'reverse triangle inequality'. The inverse image, S, of the closed set $\{1\} \subset \mathbb{R}$ is therefore closed.

Now, the standard result on metric spaces is that a subset is compact if and only if every sequence with values in the subset has a convergent subsequence with limit in the subset (if you drop the last condition then the closure is compact).

In this case we consider the sequence (of sequences)

$$a_i^{(n)} = \begin{cases} 0 & i \neq n \\ 1 & i = n \end{cases}. \tag{2}$$

This has the property that $\|a^{(n)} - a^{(m)}\|_p = 2^{\frac{1}{p}}$ whenever $n \neq m$. Thus, it cannot have any Cauchy subsequence, and hence cannot have a convergent subsequence, so S is not compact.

This is important. In fact it is a major difference between finite-dimensional and infinite-dimensional normed spaces. In the latter case the unit sphere cannot be compact whereas in the former it is.

4. Problem 1.4

Now define l^∞ as the space of bounded sequences of complex numbers with the supremum norm,

$$\|b\|_\infty = \sup_n |b_n|, \ b = (b_1, b_2, \dots), \ b_n \in \mathbb{C}. \tag{1}$$

Show that each element of l^∞ defines a continuous linear function(al) on l^1 by 'pairing'

$$F_b(a) = \sum_n b_n a_n, \ a \in l^1, \ b \in l^\infty \tag{2}$$

Solution: First of all, the series $\sum_n b_n a_n$ converges absolutely: it is majorized by the convergent series $\sum_n \|b\|_\infty |a_n| = \|b\|_\infty \cdot \|a\|_1$. The linearity of $F_b(\cdot)$ is also straightforward:

$$F_b(\lambda a + \lambda' a') = \sum_n (\lambda b_n a_n + \lambda' b_n a'_n) = \lambda \sum_n b_n a_n + \lambda' \sum_n b_n a'_n = \lambda F_b(a) + \lambda' F_b(a').$$

We're left with the continuity; recall that for linear functionals this is equivalent to finding a constant C (perhaps depending on b) such that $|F_b(a)| \leq C\|a\|_1$ for all $a \in \ell^1$. Indeed, we have

$$|F_b(a)| \leq \sum_n |b_n||a_n| \leq \|b\|_\infty \cdot \|a\|_1.$$

5. Problem 1.5

Consider the function

$$\|\cdot\| : \mathbb{C}^2 \longrightarrow [0, \infty),\ \|v\| = |v_2|,\ v = (v_1, v_2). \tag{1}$$

Show that this is a seminorm. Find the subset $S \subset \mathbb{C}^2$ on which $\|\cdot\|$ vanishes and show that $W = \mathbb{C}^2/S$ is a linear space on which $\|\cdot\|$ defines a norm.

Solution: The triangle inequality on $\mathbb{C}$ shows this is a seminorm – it is non-negative, $\|cv\| = |cv_2| = |c||v_2| = |c|\|v\|$, and

$$\|(v_1, v_2)+(w_2, w_2)\| = \|(v_1+v_2, w_1, w_2)\| = |v_2+w_2| \le |v_2|+|w_2| = \|(v_1, v_2)\|+\|(w_1, w_2)\|. \tag{2}$$

If $\|v\| = 0$ then $v_2 = 0$ and conversely so the subset S is therefore the linear subspace $\{(v_1, v_2)\} \subset \mathbb{C}^2$.

The space $\mathbb{C}^2/S$ is linearly identified with $\mathbb{C}$ by mapping $v_2 \in \mathbb{C}$ to the equivalence class, $\{(z, v_2); z \in \mathbb{C}\}$ of $(0, v_2) \in \mathbb{C}/S$ and the seminorm is identified with the norm on $\mathbb{C}$.

6. Problem 1.6[Extra]

Show that l^∞ is the dual space of l^1, namely that every bounded linear functional on l^1 is given by pairing with a unique element of l^∞.

Solution: Given a bounded linear functional $F : \ell^1 \to \mathbb{C}$, we'd like to find a bounded sequence $b = (b_1, b_2, \ldots) \in \ell^\infty$ such that $F = F_b$, in the notation of Problem 1.4. It is easy to guess what the candidate for b should look like; for a functional F_b the i-th term of the defining sequence can be reconstructed as $F_b(e_i)$, where e_i denotes the sequence with the only 1 in the i-th position and the rest filled with zeroes.

So, start with a bounded functional F, such that $|F(a)| \le C\|a\|_1$ for all $a \in \ell^1$. Consider the sequence $b = (F(e_1), F(e_2), \ldots)$. It belongs to ℓ^∞, since $|F(e_i)| \le C\|e_i\|_1 = C$. Now for every $a = (a_1, a_2, \ldots) \in \ell^1$ one has

$$\begin{aligned} F(a) &= F\left(\lim_{N\to\infty} \sum_{n=1}^{N} a_n e_n\right) \\ &\overset{\text{contin.}}{=} \lim_{N\to\infty} F\left(\sum_{n=1}^{N} a_n e_n\right) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} a_n F(e_n) \\ &= \sum_{n=1}^{\infty} a_n F(e_n), \end{aligned}$$

so F is indeed given by pairing with b.

7. Problem 1.7[Extra]

Construct a non-continuous linear functional on a normed space.

Discussion. This is pretty easy if you don't demand that the normed space be complete. Take for instance the linear space of all terminating sequences of complex numbers – so each element is a finite sequence, or arbitrary length, followed by zeros. This is a subspace of each of the l^p spaces and is dense if $p < \infty$. Now, take for instance the l^2 norm, which gives you a normed space – not complete of course. Then take as linear functional the sum of the terms of the sequence. This can be seen NOT to be bounded with respect to the l^2 norm.

Now, if you want to do this for a Banach space – and I did not ask that – then it is much harder work.

Solution: Following the hints above, define L to be the space of terminating sequences $(a_i \in \mathbb{C})_{i=1}^{\infty}$ with the ℓ^2-norm. The functional

$$\begin{aligned} F : L &\to \mathbb{C} \\ (a_i)_{i=1}^{\infty} &\mapsto \sum_i a_i \end{aligned}$$

is clearly well-defined and linear. Suppose that F is bounded, that is $|F(a)| \leq C\|a\|_2$, and plug in the sequence $a^{(n)} = (\underbrace{1, \ldots, 1}_{n \text{ units}}, 0, 0, \ldots)$:

$$n = |F(a^{(n)})| \leq C\|a^{(n)}\|_2 = C\sqrt{n}.$$

This gives a contradiction for large n.

Department of Mathematics, Massachusetts Institute of Technology
Email address: `rbm@math.mit.edu`