SECOND TEST IN 18.102, 18 APRIL, 2013
SOLUTIONS (SOMEWHAT BRIEF)

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Question 1

Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H \tag{1}$$

where v is the weak limit.

Solution. If $v_n \to v$ then by the continuity of the norm, $\|v_n\| \to \|v\|$. Conversely, suppose $v_n \rightharpoonup v$ and $\|v_n\| \to \|v\|$. For any weakly convergent sequence in a separable Hilbert space $\|v\| \le \liminf \|v_n\|$ so given $\epsilon > 0$, $\|v\| \le \|v_n\| + \epsilon$ for large n. Since $v_n + v \rightharpoonup 2v$ the parallelogram law gives

$$\|v - v_n\|^2 = 2\|v\|^2 + 2\|v_n\|^2 - \|v + v_n\|^2 \le 2\|v\|^2 + 2\|v_n\|^2 - 4\|v\|^2 + \epsilon$$

for large n, so $v \to v_n$.

Question 2

Let e_k, $k \in \mathbb{N}$, be an orthonormal basis in a separable Hilbert space, H. Show that there is a uniquely defined bounded linear operator $T : H \longrightarrow H$, satisfying

$$Te_j = e_{j-1}\ \forall\ j \ge 2,\ Te_1 = 0, \tag{2}$$

and that $T + B$ has one-dimensional null space if B is bounded and $\|B\| < 1$.

Solution: Define $Tv = \sum\limits_{j\ge 2} \langle v, e_j\rangle e_{j-1}$ for all $v \in H$. The (2) holds, T is linear and $\|Tv\| \le \|v\|$ by Bessel's inequality. Similarly we define $Sv = \sum\limits_{j\ge 1} \langle v, e_j\rangle e_{j+1}$ and note that $\|S\| \le 1$, $TS = \operatorname{Id}$. If $B : H \longrightarrow H$ is a bounded linear operator with $\|B\| < 1$ we look for a solution of $(T + B)v = 0$ in the form $v = e_1 + Sw$ which is non-zero since $\langle e_1, Sw\rangle = 0$. Thus

$$(T + B)(e_1 + Sw) = Te_1 + TSw + Be_1 + BSw = 0 \text{ iff } (\operatorname{Id} + BS)w = -Be_1.$$

Since $\|BS\| \le \|B\|\|S\|$ we know from Neumann series that $\operatorname{Id} + BS$ is invertible, so such a w exists and hence $T + B$ has null space of dimension at least one. However, the argument can be reversed to see that the only elements in the null space are of the $c(e_1 + Sw)$ for the $w = -(\operatorname{Id} + BS)^{-1}Be_1$ constructed above.

Question 3

Show that a continuous function $K : [0,1] \longrightarrow L^2(0,2\pi)$ has the property that the Fourier series of $K(x) \in L^2(0,2\pi)$, for $x \in [0,1]$, converges uniformly in the sense that if $K_n(x)$ is the sum of the Fourier series over $|k| \le n$ then $K_n : [0,1] \longrightarrow L^2(0,2\pi)$ is also continuous and

$$\sup_{x\in[0,1]} \|K(x) - K_n(x)\|_{L^2(0,2\pi)} \to 0. \tag{3}$$

Solution: Since K is a compact metric space, the image of $[0,1]$ under a continuous map into the metric space $L^2(0,2\pi)$ is a compact set. The equi-small tails property of compact sets implies that if P_n is the projection onto the span of the terms e_k, $|k| \le n$, in the Fourier basis then given $\epsilon > 0$ there exists n such that

$$\|(\operatorname{Id} - P_n)K(x)\|_{L^2} \le \epsilon \ \forall\ x \in [0,1].$$

Now, $K_n(x) = P_nK(x) = \sum_{|k|\le n} \langle K(x), e_k\rangle e_k$ is continuous on $[0,1] \times [0,2\pi]$, since the coefficients $\langle K(x), e_k\rangle$ are continuous in x (the inner product being continuous on L^2) and

$$\sup_{x\in[0,1]} \|K(x) - K_n(x)\|_{L^2} = \|(\operatorname{Id} - P_n)K(x)\|_{L^2} < \epsilon$$

for large n show the convergence to 0.

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