PROBLEM SET 5 FOR 18.102, SPRING 2013 DUE THURSDAY MARCH 29 (SO 5AM SATURDAY MARCH 30).

RICHARD MELROSE

Since it will not be returned until after Spring Break you can have until March 29 to complete this problem set. You are of course welcome to send it in early.

Problem 5.1

Show that for any $f \in L^2(\mathbb{R})$ the zero extension outside an interval, $\chi_{[-R,R]}f \in L^2(\mathbb{R})$ and that

$$\lim_{R\to\infty} \|f - \chi_{[-R,R]}f\|_{L^2} = 0. \tag{5.1}$$

Problem 5.2

Show that any element of $L^2(\mathbb{R})$ is continunous-in-the-L^2-mean in the sense that

$$\lim_{t\to 0} \int |f(\cdot) - f(\cdot - t)|^2 = 0 \tag{5.2}$$

[including that the norm is well-defined].

Problem 5.3

Show that $S \subset L^2(\mathbb{R})$ is compact if and only if it is closed, bounded and both 'uniformly small at infinity' and 'uniformly continous in the mean' [where you need to properly formulate these conditions].

Okay, so the sufficiency of the two 'equi-' conditions is maybe a bit hard without some guidance. Fortunately Charles handed in his homework early so I follow his idea, with some variation.

We want to show that a set which is closed and bounded and satisfies them is compact. Thus we need to show that any sequence in it has a convergent subsequence. We can pass to a weakly convergent sequence using boundedness and all we need to show is that if IT satisfies your two equi- conditions then it is actually strongly convergent. If f_n is the sequence and f the weak limit you can easily check that $g_n = f_n - f$ which is weakly convergent to zero also satisfies the two equi-conditions. If $\|g_n\| \to 0$ we are done, so we can pass to a subsequence and suppose that $\|g_n\| > \delta > 0$. In fact, we may as well suppose that $\|g_n\| = 1$ since dividing by the norms still gives us an equi- sequence with weak limit 0. It is enough to show that for any $\epsilon > 0$ there is a strongly convergent sequence close by, $h_n \to h$, $\|g_n - h_n\| < \epsilon$ (since this gives the contradiction that the weak limit is non-zero). Okay, now replace g_n by $g'_n = \chi_{[-R,R]}g_n$ with R chosen so large that this is close to g_n – using one equi- property. If you are more comfortable with it, you can replace

g_n' by a sequence of continous functions of compact support in $[-R-1, R+1]$ keeping equi-continuity-in-the-mean and making a very small, uniform error. So, here is where I leave you to work. If you cut $[-R-1, R+1]$ up into a sufficiently large number of equal parts and replace g_n' by the new sequence G_n of step functions where the value of G_n on each of these intervals is the mean of g_n' on it, then by the equi- condition $\|g_n' - G_n\|_{L^2}$ is uniformly small. Finally then you have a sequence in a finite dimensional space – if you trace back you will see it is close to the original and weakly convergent, hence strongly convergent. Or just use Heine-Borel to pass to a strongly convergent subsequence and you still get a contradiction.

Maybe someone will find an even easier way. The traditional approach is via convolution, which we should get to later.

Problem 5.4

Work out the Fourier coefficients $c_k(t) = \int_{(0,2\pi)} f_t e^{-ikx}$ of the step function

$$f_t(x) = \begin{cases} 1 & 0 \le x < t \\ 0 & t \le x \le 2\pi \end{cases} \tag{5.3}$$

for each fixed $t \in (0, 2\pi)$.

Problem 5.5

Give an example of a closed subset of a Hilbert space which is not weakly closed – which contains a weakly convergent subsequence which has weak limit not in the set.

Problem 5.6 – Extra

Now, suppose that you know that the Fourier basis $e^{ikx}/\sqrt{2\pi}$ is complete in $L^2(0, 2\pi)$. Use this to prove that for appropriate constants d_k, the functions $d_k \sin(kx/2)$, $k \in \mathbb{N}$, form an orthonormal basis for $L^2(0, 2\pi)$. (Hint think of extending functions to $(-2\pi, 2\pi)$ to be odd, use the corresponding Fourier basis and see what this means).

Problem 5.7 – Extra

At this stage we have NOT proved that the Fourier functions $e^{ikx}/\sqrt{2\pi}$ form an orthonormal basis – we have not shown they are complete. So, without assuming this explain why the Fourier series in Problem 5.4 converges to f_t in $L^2(0, 2\pi)$ if and only if

$$2 \sum_{k>0} |c_k(t)|^2 = 2\pi t - t^2, \ t \in (0, 2\pi). \tag{5.4}$$

Write the condition (5.4) out as a Fourier series and apply the argument again to show that the completeness of the Fourier basis implies identities for the sum of k^{-2} and k^{-4}.

Can you explain how reversing the argument, that knowledge of the sums of these two series might imply the completeness of the Fourier basis?

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: `rbm@math.mit.edu`