PROBLEM SET 4 FOR 18.102, SPRING 2013
DUE FRIDAY 15 MARCH (SO 4AM, 16 MARCH).

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Problem 4.1

Let H be a normed space (over $\mathbb{C}$) in which the norm satisfies the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)\ \forall\ u, v \in H. \tag{1}$$

Show that

$$(u,v) = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\right) \tag{2}$$

is a positive-definite Hermitian form which induces the given norm.

Problem 4.2

Let H be a finite dimensional (pre)Hilbert space. So, by definition H has a basis $\{v_i\}_{i=1}^n$, meaning that any element of H can be written

$$v = \sum_i c_i v_i \tag{3}$$

and there is no dependence relation between the v_i's – the presentation of $v = 0$ in the form (3) is unique. Show that H has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying $(e_i, e_j) = \delta_{ij}$ ($= 1$ if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (3) are $c_i = (v, e_i)$ and that the map

$$T : H \ni v \longmapsto ((v, e_1), (v, e_2), \dots, (v, e_n)) \in \mathbb{C}^n \tag{4}$$

is a linear isomorphism with the properties

$$(u, v) = \sum_i (Tu)_i \overline{(Tv)_i},\ \|u\|_H = \|Tu\|_{\mathbb{C}^n}\ \forall\ u, v \in H. \tag{5}$$

Why is a finite dimensional preHilbert space a Hilbert space?

Problem 4.3

Let e_i, $i \in \mathbb{N}$, be an orthonormal sequence in a separable Hilbert space H. Suppose that for each element u in a dense subset $D \subset H$

$$\sum_i |(u, e_i)|^2 = \|u\|^2. \tag{6}$$

Conclude that e_i is an orthonormal basis, i.e. is complete.

Problem 4.4

Consider the sequence space

$$h^{2,1} = \left\{ c : \mathbb{N} \ni j \longmapsto c_j \in \mathbb{C}; \sum_j (1+j^2)|c_j|^2 < \infty \right\}. \tag{7}$$

(1) Show that

$$h^{2,1} \times h^{2,1} \ni (c,d) \longmapsto \langle c, d\rangle = \sum_j (1+j^2) c_j \overline{d_j} \tag{8}$$

is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.

(2) Denoting the norm on this space by $\|\cdot\|_{2,1}$ and the norm on l^2 by $\|\cdot\|_2$, show that

$$h^{2,1} \subset l^2,\ \|c\|_2 \le \|c\|_{2,1}\ \forall\ c \in h^{2,1}. \tag{9}$$

Problem 4.5

Suppose that H_1 and H_2 are two different Hilbert spaces and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint) $A^* : H_2 \longrightarrow H_1$ with the property

$$\langle Au_1, u_2\rangle_{H_2} = \langle u_1, A^*u_2\rangle_{H_1}\ \forall\ u_1 \in H_1,\ u_2 \in H_2. \tag{10}$$

Problem 4.6 – Extra

Recall (from Rudin's book for instance) that if $F : [a,b] \longrightarrow [A,B]$ is an increasing continuously differentiable map, in the strong sense that $F'(x) > 0$, between finite intervals then for any continuous function $f : [A,B] \longrightarrow \mathbb{C}$, (Rudin shows it for Riemann integrable functions)

$$\int_A^B f(y)dy = \int_a^b f(F(x))F'(x)dx. \tag{11}$$

Prove the same identity for every $f \in \mathcal{L}^1((A,B))$, which in particular requires the right side to make sense.

Problem 4.7 – Extra

A subset $E \subset \mathbb{R}$ is said to be of finite measure (resp. measurable) if the characteristic function

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \tag{12}$$

is in $\mathcal{L}^1(\mathbb{R})$ (resp. $\chi_{[-R,R]}\chi_E \in \mathcal{L}^1(\mathbb{R})$ for every R). The measure is

$$\mu(E) = \lim_{N\to\infty} \int \chi_{[-N,N]}\chi_E$$

– so the measure of a measurable set might be infinite. Show that if E_i, $i \in \mathbb{N}$ is a countable collection of measurable sets then $E = \sum\limits_i E_i$ is measurable and that

$$\begin{gathered} \mu(E) \le \sum_i \mu(E_i), \\ \mu(E) = \sum_i \mu(E_i) \text{ if } E_i \cap E_j = \emptyset \text{ for } i \ne j. \end{gathered} \tag{13}$$

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