PROBLEM SET 2 FOR 18.102, SPRING 2013
DUE 4AM (DOESN'T SEEM TO WORK) SATURDAY 23 FEB.

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Here is what I said before about collaboration and all that. I do not mind who you talk to, what you read or where you find information (for this problem in some cases there are not solutions in the notes). However, I expect that you will devise and write out the answers yourself. This means precisely no direct copying, you must first assimilate the material then rewrite it.

Again the homework consists of seven problems of which you should do five. You can get full marks by doing any five of them, probably the first five are the most straightforward and you cannot get more than 50 marks – only your 'best' five solutions will count.

1. Problem 2.1

Suppose $a < b$ are real, show that the step function

$$\chi_{(a,b]} = \begin{cases} 0 & \text{if } x \le a \\ 1 & \text{if } a < x \le b \\ 0 & \text{if } b < x \end{cases} \tag{1}$$

is an element of $\mathcal{L}^1(\mathbb{R})$. [Note that the definition requires you to find an absolutely summable series of continuous functions with appropriate properties.]

2. Problem 2.2

A subset $E \subset \mathbb{R}$ is said to be *of measure zero* if there exists an absolutely summable sequence $f_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ such that

$$E \subset \{x \in \mathbb{R}; \sum_n |f_n(x)| = +\infty\}. \tag{1}$$

Show that if E is of measure zero and $\epsilon > 0$ is given then there exists $f_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ satisfying (1) and in addition

$$\sum_n \int |f_n| < \epsilon. \tag{2}$$

3. Problem 2.3

Using the previous problem (or otherwise ...) show that a countable union of sets of measure zero is a set of measure zero.

4. Problem 2.4

Suppose $g_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ is a sequence of non-negative functions such that $g_n(x)$ is a non-increasing sequence for each $x \in \mathbb{R}$. Show that the limiting function $g(x) = \lim_{n\to\infty} g_n(x)$ is Lebesgue integrable.

5. Problem 2.5

Suppose $h : \mathbb{R} \longrightarrow \mathbb{R}$ is an element of $\mathcal{L}^1(\mathbb{R})$. Write out, starting from the definition, a proof that the positive part of h :

$$h_+(x) = \begin{cases} h(x) & \text{if } h(x) \geq 0 \\ 0 & \text{if } h(x) < 0 \end{cases} \tag{1}$$

is an element of $\mathcal{L}^1(\mathbb{R})$ and that so is the negative part.

6. Problem 2.6 – Extra

Let's generalize the theorem about $\mathcal{B}(V, W)$ to bilinear maps – this may seem hard but just take it step by step!

(1) Check that if U and V are normed spaces then $U \times V$ (the linear space of all pairs (u, v) where $u \in U$ and $v \in V$) is a normed space where addition and scalar multiplication is 'componentwise' and the norm is the sum

$$\|(u, v)\|_{U\times V} = \|u\|_U + \|v\|_V. \tag{1}$$

(2) Show that $U \times V$ is a Banach space if both U and V are Banach spaces.

(3) Consider three normed spaces U, V and W. Let

$$B : U \times V \longrightarrow W \tag{2}$$

be a *bilinear* map. This means that

$$B(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 B(u_1, v) + \lambda_2 B(u_2, v),$$
$$B(u, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 B(u, v_1) + \lambda_2 B(u, v_2)$$

for all u, u_1, $u_2 \in U$, v, v_1, $v_2 \in V$ and λ_1, $\lambda_2 \in \mathbb{C}$. Show that B is continuous if and only if it satisfies

$$\|B(u, v)\|_W \leq C\|u\|_U \|v\|_V \ \forall\ u \in U,\ v \in V. \tag{3}$$

(4) Let $\mathcal{M}(U, V; W)$ be the space of all such continuous bilinear maps. Show that this is a linear space and that

$$\|B\| = \sup_{\|u\|=1, \|v\|=1} \|B(u, v)\|_W \tag{4}$$

is a norm.

(5) Show that $\mathcal{M}(U, V; W)$ is a Banach space if W is a Banach space.

7. Problem 2.7 – Extra

Consider the space $\mathcal{C}_c(\mathbb{R}^n)$ of continuous functions $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ which vanish outside a compact set, i.e. in $|x| > R$ for some R (depending on u). Check (quickly) that this is a linear space.

Show that if $y \in \mathbb{R}^{n-1}$ and $u \in \mathcal{C}^0_c(\mathbb{R}^n)$ then

$$U_y : \mathbb{R} \ni t \longmapsto u(y, t) \in \mathbb{C} \tag{1}$$

defines an element $U_y \in \mathcal{C}^0_c(\mathbb{R})$. Show further that $\mathbb{R}^{n-1} \ni y \longmapsto U_y$ is a continuous map into $\mathcal{C}^0_c(\mathbb{R})$ *with respect to the supremum norm* which vanishes for $|y| > r$, i.e. has compact support. Conclude that 'integration in the last variable' gives a continuous linear map (with respect to supremum norms)

$$\mathcal{C}_c(\mathbb{R}^n) \ni u \longrightarrow v \in \mathcal{C}_c(\mathbb{R}^{n-1}),\ v(y) = \int U_y. \tag{2}$$

By iterating this statement show that the iterated Riemann integral is well defined

$$\int : \mathcal{C}_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{3}$$

and that $\int |u|$ is a norm.

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