PROBLEM SET 1 FOR 18.102, SPRING 2013 DUE, ELECTRONICALLY ONLY, TO THE EMAIL ADDRESS BELOW BY 4AM SATURDAY 16 FEB.

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About collaboration and all that. I do not mind who you talk to, what you read or where you find information (for most of the problems the solutions are in the notes). However, I expect that you will devise and write out the answers yourself. This means precisely no direct copying, you must first assimilate the material then rewrite it.

Note that it is okay to write the solutions out by hand and then scan-to-pdf or some such. Just make sure the result is readable.

The homework consists of five problems. You should do the first five and then if you want a little more of a challenge try the additional problem – however you cannot get more than 50 marks.

1. Problem 1.1

Show that the norm on any normed space is a continuous function.

2. Problem 1.2

Write out a proof for each p with $1 \le p < \infty$ that

$$l^p = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty,\ a_j = a(j)\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold. Note that the only 'tricky' part is the triangle inequality for this all you really need in the way of 'hard estimates' is to show that (for all N)

$$\left(\sum_{j=1}^{N} |a_j|^p\right)^{\frac{1}{p}} \text{ is a norm on } \mathbb{C}^N.$$

I'm expecting that you will look up and give a brief proof of (2).

3. Problem 1.3

Prove directly that each l^p as defined in Problem 1.1 is a Banach space.

Remarks (for those who need orientation): This means showing that each Cauchy sequence converges. The problem here is to find the limit of a given Cauchy sequence. The usual approach is show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements (which are sequences) at the Nth term gives a Cauchy sequence with respect to the norm coming from (2) on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it really is the limit.

4. Problem 1.4

Consider the 'unit sphere' in l^p. This is the set of vectors of length 1 :

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) Show that S is closed.
(2) Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e.g. by checking in Rudin).
(3) Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a 1 in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!

5. Problrm 1.5

Now define l^∞ as the space of bounded sequences of complex numbers with the supremum norm,

$$\|b\|_\infty = \sup_n |b_n|,\ b = (b_1, b_2, \dots),\ b_n \in \mathbb{C}. \tag{1}$$

Show that each element of l^∞ defines a continuous linear function(al) on l^1 by 'pairing'

$$F_b(a) = \sum_n b_n a_n,\ a \in l^1,\ b \in l^\infty \tag{2}$$

6. Problem 1.6[Extra]

Show that l^∞ is the dual space of l^1, namely that every bounded linear functional on l^1 is given by pairing with a unique element of l^∞.

7. Problem 1.7[Extra]

Construct a non-continuous linear functional on a normed space.

Discussion. This is pretty easy if you don't demand that the normed space be complete. Take for instance the linear space of all terminating sequences of complex numbers – so each element is a finite sequence, or arbitrary length, followed by zeros. This is a subspace of each of the l^p spaces and is dense if $p < \infty$. Now, take for instance the l^2 norm, which gives you a normed space – not complete of course. Then take as linear functional the sum of the terms of the sequence. This can be seen NOT to be bounded with respect to the l^2 norm.

Now, if you want to do this for a Banach space then it is much harder work.

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