

QUESTIONS 18.102 FINAL, SPRING 2013

I expect to choose 6 of these questions on the final. Note that some are rather straightforward and some are less so, I will choose some of each. Before the day of the exam, you can ask me for hints.

All Hilbert spaces should be taken to be separable.

PROBLEM 1

Let H_i , $i = 1, 2$ be two Hilbert spaces with inner products $(\cdot, \cdot)_i$ and suppose that $I : H_1 \rightarrow H_2$ is a continuous linear map between them. Suppose that

- the range of I is dense in H_2
- I is injective.
- if $v_n \rightarrow v$ (converges weakly) in H_1 then $I(v_n) \rightarrow I(v)$ in H_2 .

Show that

- (1) there is a continuous linear map $Q : H_2 \rightarrow H_1$ such that $(u, I(f))_2 = (Qu, f)_1 \forall f \in H_1$.
- (2) as a map from H_1 to itself, $Q \circ I$ is bounded and self-adjoint.
- (3) as a map on H_1 , $Q \circ I$ is compact.
- (4) as a map on H_2 , $I \circ Q$ has the same eigenvalues as $Q \circ I$ on H_1 .

PROBLEM 2

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, χ_j of A_j , is integrable for each j . Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

PROBLEM 3

If H is a Hilbert space let $l^2(H)$ be the space of sequences $h : \mathbb{N} \rightarrow H$ such that $\sum_j \|u(j)\|_H^2 < \infty$. Show that this is a Hilbert space and that there is a bounded linear bijection $l^2(H) \rightarrow H$ if and only if H is not finite dimensional.

PROBLEM 4

Let A be a Hilbert-Schmidt operator on a separable Hilbert space H , which means that for some orthonormal basis $\{e_i\}$

$$(1) \quad \sum_i \|Ae_i\|^2 < \infty.$$

Using Bessel's identity to expand $\|Ae_i\|^2$ with respect to another orthonormal basis $\{f_j\}$ show that $\sum_j \|A^*f_j\|^2 = \sum_i \|Ae_i\|^2$. Conclude that the sum in (1) is independent of the orthonormal basis used to define it and that the Hilbert-Schmidt operators form a Hilbert space.

PROBLEM 5

Let $a : [0, 2\pi] \rightarrow L^2(0, 2\pi)$ be a continuous map. Show that

$$Af(x) = \int_0^{2\pi} a(x)f, \quad f \in L^2(0, 2\pi)$$

defines a compact operator on $L^2(0, 2\pi)$. Conversely show that if $A : L^2(0, 2\pi) \rightarrow \mathcal{C}([0, 2\pi])$ is given as a bounded operator for the supremum norm on $\mathcal{C}([0, 2\pi])$ then there exists such a map a .

PROBLEM 6

Let $u_n : [0, 2\pi] \rightarrow \mathbb{C}$ be a sequence of continuously differentiable functions which is uniformly bounded, with bounded derivatives i.e. $\sup_n \sup_{x \in [0, 2\pi]} |u_n(x)| < \infty$ and $\sup_n \sup_{x \in [0, 2\pi]} |u'_n(x)| < \infty$. Show that u_n has a subsequence which converges in $L^2([0, 2\pi])$.

PROBLEM 7

Suppose that $f \in \mathcal{L}^1(0, 2\pi)$ is such that the constants $c_k = \int_{(0, 2\pi)} f(x)e^{-ikx}$, $k \in \mathbb{Z}$, satisfy $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$. Show that $f \in L^2(0, 2\pi)$.

PROBLEM 8

Carefully justify each step in the following proof of the uniform boundedness principle (see Alan D. Sokal, 'A really simple elementary proof of the uniform boundedness theorem', Arxiv math:1005.1585).

Theorem: If a sequence, or more generally a collection \mathcal{T} , of bounded operators $T : V \rightarrow W$, where V is a Banach space and W is a normed space, is such that for each $v \in V$, $\sup_{T \in \mathcal{T}} \|T(v)\|_W < \infty$ then $\sup_{T \in \mathcal{T}} \|T\| < \infty$.

Proof:

- (1) Suppose to the contrary that $\sup_{T \in \mathcal{T}} \|T\| = \infty$.
- (2) Choose a sequence T_k in \mathcal{T} such that $\|T_k\| \geq 4^k$.
- (3) Observe that for a bounded operator, S , between two normed spaces

$$\max[\|S(v+v')\|, \|S(v-v')\|] \geq \frac{1}{2} [\|S(v+v')\| + \|S(v-v')\|] \geq \|Sv'\|$$

- (4) Deduce that

$$\sup_{u \in B(v,r)} \|Su\| \geq \|S\|r \quad \forall v \in V \text{ and } r > 0.$$

- (5) Set $v_0 = 0$ and note that for $k \geq 1$, proceeding inductively, points $v_k \in V$ can be chosen such that $\|v_k - v_{k-1}\| \leq 3^{-k}$ and $\|T_k v_k\| \geq \frac{2}{3} 3^{-k} \|T_k\|$.
- (6) The sequence $\{v_k\}_{k=1}^{\infty}$ is Cauchy, hence converges to some $v \in V$.
- (7) Since $\|v - v_k\| \leq \frac{1}{2} 3^{-k}$, $\|T_k v\| \geq \frac{1}{6} 3^{-k} \|T_k\| \geq \frac{1}{6} (4/3)^k \rightarrow \infty$.
- (8) Hence $\sup_{T \in \mathcal{T}} \|T\| < \infty$.

PROBLEM 9

Let B_n be a sequence of bounded linear operators on a Hilbert space H such that for each u and $v \in H$ the sequence $(B_n u, v)$ converges in \mathbb{C} . Show that there is a uniquely defined bounded operator B on H such that

$$(Bu, v) = \lim_{n \rightarrow \infty} (B_n u, v) \quad \forall u, v \in H.$$

PROBLEM 10

Let $T : H_1 \rightarrow H_2$ be a continuous linear map between two Hilbert spaces and suppose that T is both surjective and injective.

- (1) Let $A_2 \in \mathcal{K}(H_2)$ be a compact linear operator on H_2 , show that there is a compact linear operator $A_1 \in \mathcal{K}(H_1)$ such that

$$A_2 T = T A_1.$$

- (2) If A_2 is self-adjoint (as well as being compact) and H_1 is infinite dimensional, show that A_1 has an infinite number of linearly independent eigenvectors.

PROBLEM 11

Suppose $P \subset H$ is a closed linear subspace of a Hilbert space, with $\pi_P : H \rightarrow P$ the orthogonal projection onto P . If H is separable and A is a compact self-adjoint operator on H , show that there is a complete orthonormal basis of H each element of which satisfies $\pi_P A \pi_P e_i = t_i e_i$ for some $t_i \in \mathbb{R}$.

PROBLEM 12

Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, where $j = 1, 2, \dots$, and $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. You may assume completeness in $L^2(\mathbb{R})$ and use the facts established in class that $-\frac{d^2 e_j}{dx^2} + x^2 e_j = (2j+1)e_j$, that $c_j = 2^{-j/2} (j!)^{-\frac{1}{2}} \pi^{-\frac{1}{4}}$ and that $e_j = p_j(x) e_0$ for a polynomial of degree j . Compute $C e_j$ and $A e_j$ in terms of the basis and hence arrive at a formula for de_j/dx . Use this to show that the sequence $j^{-\frac{1}{2}} \frac{de_j}{dx}$ is bounded in $L^2(\mathbb{R})$. Conclude that if

$$(2) \quad H_{\text{iso}}^1 = \left\{ u \in L^2(\mathbb{R}); \sum_{j \geq 1} j |(u, e_j)|^2 < \infty \right\}$$

then there is a uniquely defined operator $D : H_{\text{iso}}^1 \rightarrow L^2(\mathbb{R})$ such that $D e_j = \frac{de_j}{dx}$ for each j .

PROBLEM 13

Let A be a compact self-adjoint operator on a separable Hilbert space and suppose that for *any* orthonormal basis

$$\sum_i |(A e_i, e_i)| < \infty.$$

Show that the eigenvalues of A , if infinite in number, form a sequence in l^1 .

PROBLEM 14

Consider the subspace $H \subset C[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

$$(3) \quad u(x) = \int_0^x U(x), \quad \forall x \in [0, 2\pi]$$

for some $U \in L^2(0, 2\pi)$ (depending on u of course). Show that the function U is determined by u (given that it exists), that

$$(4) \quad \|u\|_H^2 = \int_{(0, 2\pi)} |U|^2$$

turns H into a Hilbert space.

PROBLEM 15

Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, where $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. Define an operator on $L^2(\mathbb{R})$ by

$$Au = \sum_{j=0}^{\infty} (2j+1)^{-\frac{1}{2}} (u, e_j)_{L^2} e_j.$$

- (1) Show that A is compact as an operator on $L^2(\mathbb{R})$.
- (2) Suppose that $V \in C_{\infty}^0(\mathbb{R})$ is a bounded, real-valued, continuous function on \mathbb{R} . Show that $L^2(\mathbb{R})$ has an orthonormal basis consisting of eigenfunctions of $K = AVA$, where V is acting by multiplication on $L^2(\mathbb{R})$.

PROBLEM 16

Suppose that $f \in \mathcal{L}^1(0, 2\pi)$ is such that the constants

$$c_k = \int_{(0, 2\pi)} f(x) e^{-ikx}, \quad k \in \mathbb{Z},$$

satisfy

$$\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty.$$

Show that $f \in \mathcal{L}^2(0, 2\pi)$.

PROBLEM 17

Consider the space of those complex-valued functions on $[0, 1]$ for which there is a constant C (depending on the function) such that

$$(5) \quad |u(x) - u(y)| \leq C|x - y|^{\frac{1}{2}} \quad \forall x, y \in [0, 1].$$

Show that this is a Banach space with norm

$$(6) \quad \|u\|_{\frac{1}{2}} = \sup_{[0, 2\pi]} |u(x)| + \inf_{(5) \text{ holds}} C.$$

PROBLEM 18

Let $B : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow \mathbb{C}$ be a bilinear form (meaning it is linear in each factor when the other is held fixed) such that there is a constant $C > 0$ and

$$|B(u, v)| \leq C \|u\|_{L^2} \|v\|_{L^2} \quad \forall u, v \in L^2(\mathbb{R}).$$

Show that there is a bounded linear operator $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} T(u)(x)v(x) = B(u, v) \quad \forall u, v \in L^2(\mathbb{R}).$$