NOTES FOR 18.102, 25 APRIL, 2013

Let's try to understand a little more based on what I did last time. We can define the 'Dirichlet domain' to be

$$
\begin{aligned}
&H^2_D(0,2\pi) = \{(u,f) \in L^2(0,2\pi)\times L^2(0,2\pi);\\
(1)\qquad &\exists\ v_n : [0,2\pi] \longrightarrow \mathbb{C},\ \text{twice continuously differentiable}\\
&\text{with } v_n(0) = 0 = v_n(2\pi),\ v_n \to u \in L^2(0,2\pi),\ -\frac{d^2v_n}{dx^2} \to f \in L^2(0,2\pi)\}.
\end{aligned}
$$

This has a natural preHilbert structure given by

$$((u_1,f_1),(u_2,f_2))_{H^2_D} = \int u_1\overline{u}_2 + \int f_1\overline{f_2}. \tag{2}$$

That is, we regard it as a subspace of $L^2(0,2\pi)\times L^2(0,2\pi)$.

So, apply your understanding of L^2 and elementary calculus! It is a preHilbert space, is it complete? Yes, basically because it is defined as a closure. If (u_n,f_n) is Cauchy then by completeness of L^2, $u_n \to u$ and $f_n \to f$ in $L^2(0,2\pi)$. Is the pair $(u,f) \in H^2_D(0,2\pi)$? Suppose $v_{n,j} \to u_n$ and $-\frac{d^2}{dx^2}v_{n,j} \to f_n$ in $L^2(0,2\pi)$ for each n with $v_{n,j}$ twice continuously differentiable and with $v_{n,j}(0) = 0 = v_{n,j}(2\pi)$ then for each n there exists $j = j(n)$ such that

$$\|v_{n,j} - u_n\|^2_{L^2} + \| - \frac{d^2v_{n,j}}{dx^2} - f_n\|^2 < 2^{-n}. \tag{3}$$

Then the sequence $w_n = v_{n,j(n)}$ is such that $(w_n, -\frac{d^2w_n}{dx^2}) \to (u,f)$ in $L^2(0,2\pi)\times L^2(0,2\pi)$. Thus in fact $H^2_D(0,2\pi)$ is a Hilbert space.

Last time I defined an operator K which is an integral operator

$$
\begin{aligned}
&L^2(0,2\pi) \ni f \longmapsto Kf \in L^2(0,2\pi),\ Kf(x) = \int_0^{2\pi} K(x,s)f(s)ds,\\
(4)\qquad &K(x,s) = (s-x)H(x-s) + \frac{x}{2\pi}(2\pi - s) = \begin{cases} s - x + x - \frac{xs}{2\pi} & x \ge s\\ x - \frac{xs}{2\pi} & x \le s\end{cases}\\
&\Longrightarrow K(x,s) = \min(x,s) - \frac{xs}{2\pi} \ge 0.
\end{aligned}
$$

In fact

$$Kf(x) = -\int_0^x\int_0^t f(s)dsdt + \frac{x}{2\pi}\int_0^{2\pi}\int_0^t f(s)dsdt \text{ if } f \in \mathcal{C}([0,2\pi]) \tag{5}$$

(this is where it came from). So from the Fundamental Theorem of Calculus we know:-

Proposition 1. *If $f \in \mathcal{C}([0,2\pi])$ then $u = Kf$ is twice continuously differentiable and is the unique solution of the Dirichlet problem*

$$-\frac{d^2u}{dx^2} = f \text{ on } [0,2\pi],\ u(0) = 0 = u(2\pi). \tag{6}$$

Last time I also noted that

Lemma 1. *As an operator on* $L^2(0,2\pi)$, K *is compact and self-adjoint with the* $\sqrt{\pi}^{-1}\sin kx/2$, $k\in\mathbb{N}$, *an orthonormal basis of eigenfunctions with corresponding eigenvalues* $4k^{-2}$.

Proof. The proof is to note that the kernel $K\in\mathcal{C}([0,2\pi]^2)$ and hence

$$K: L^2(0,2\pi)\longrightarrow\mathcal{C}([0,2\pi]).$$

It is self-adjoint since $K(x,s)$ is real and $K(s,x)=K(x,s)$. It follows from Proposition 1 above and the fact that $\frac{d^2}{dx^2}\sin kx/s = k^2/4\cdot\sin kx/2$ with these functions satisfying the boundary conditions so $K(\sin kx/2)=\lambda_k\sin kx/2$, $\lambda_k=4/k^2$. Thus these are indeed eigenfunctions for K and we know from Fourier series that these form a complete set of orthogonal functions in $L^2(0,2\pi)$. Thus in fact K is determined by these values on the orthonormal basis, so it is compact. □

Proposition 2. *If* $f\in L^2(0,2\pi)$ *then* $(Kf,f)\in H^2_D(0,2\pi)$ *and the map*

$$L^2(0,2\pi)\ni f\longmapsto(Kf,f)\in H^2_D(0,2\pi)\tag{7}$$

is an isomorphism, a continuous bijection.

Proof. The first part is really a corollary of the preceding Proposition. Namely, if $f\in L^2(0,2\pi)$ then we know there exists a sequence $f_n\in\mathcal{C}([0,2\pi])$ such that $f_n\to f$ in $L^2(0,2\pi)$. So consider $u_n=Kf_n$. By the Proposition this is twice continuously differentiable, has $u_n(0)=0=u_n(2\pi)$ and $-\frac{d^2u_n}{dx^2}=f_n$. Since $f_n\to f$ and K is continuous, $u_n=Kf_n\to Kf$. So by definition of the space, $(Kf,f)\in H^2_D(0,2\pi)$.

Conversely, if $(u,f)\in H^2_D(0,2\pi)$ then $f\in L^2(0,2\pi)$ and we have just shown that $(Kf,f)\in H^2_D(0,2\pi)$. So it follows that $(v,0)\in H^2_D(0,2\pi)$ where $v=Kf-u$. So we want to show that this implies $v=0$. By definition there is a sequence v_n as in (1) with

$$v_n\to v,\ w_n=-\frac{d^2v_n}{dx^2}\to 0\text{ in }L^2(0,2\pi).\tag{8}$$

Consider the expansion of v_n in the orthonormal basis $e_k=c\sin(kx/2)$, $c=1/\sqrt{\pi}$. Thus

$$v_n=\sum_{k\ge1}a_{n,k}e_k,\text{ converges in }L^2(0,2\pi).\tag{9}$$

However, $w_n\in L^2(0,2\pi)$ as well and we can see that its Fourier(-Bessel) coefficients are

$$\int_0^{2\pi}w_ne_k=-\int_0^{2\pi}\frac{d^2v_n}{dx^2}e_k=\frac{k^2}{4}\int_0^{2\pi}v_ne_k=\frac{k^2}{4}a_{n,k}.\tag{10}$$

By assumption, $w_n\to0$ in $L^2(0,2\pi)$, but this means that each of the Fourier-Bessel coefficents must tend to zero, so $\frac{k^2}{4}a_{n,k}\to0$ for each k and hence $a_{n,k}\to0$ for each k. Thus in fact $v_n\rightharpoonup0$ in $L^2(0,2\pi)$. Since $v_n\to v$, the uniqueness of weak limits implies that $v=0$ which is what we wanted to know.

Thus we have shown that $(u,f)\in H^2_D(0,2\pi)$ if and only if $u=Kf$.

Continuity follows from the definition of the norms and the boundedness of K. □

Notice that K is injective, $Kf = 0$ implies $f = 0$ – we computed the eigenvalues last time as $4/k^2$. So really we do not need the pair (u, f) to specify an element of $H^2_D(0, 2\pi)$ since if we know $u \in L^2(0, 2\pi)$ and that there exists $f \in L^2(0, 2\pi)$ such that $u = Kf$ and hence $(u, f) \in H^2_D(0, 2\pi)$ then there is only one such f.

Notation:- We identify pairs in $H^2_D(0, 2\pi)$ with their first elements and so redefine it unambiguously as

$$H^2_D(0, 2\pi) = \{u \in L^2(0, 2\pi);\ \exists\ f \in L^2(0, 2\pi),\ u = Kf\}. \tag{11}$$

The norm remains the same – it is $\|u\|^2_{H^2_D} = \|u\|^2_{L^2} + \|f\|^2_{L^2}$.

Note that the space

$$\{u : [0, 2\pi] \longrightarrow \mathbb{C};\ u \text{ is twice continuously differentiable },\\ u(0) = u(2\pi)\} \subset H^2_D(0, 2\pi) \tag{12}$$

since in this case $u = Kf$ if $f = -\frac{d^2u}{dx^2}$. This is a *dense* subspace of $H^2_D(0, 2\pi)$.

Proposition 3. *The map*

$$D^2 : H^2_D(0, 2\pi) \ni u \longmapsto f \in L^2(0, 2\pi),\ \textit{where } u = Kf \tag{13}$$

is an isomorphism of H^2_D to $L^2(0, 2\pi)$.

It is usual to write this isomorphism as $D^2 = -\frac{d^2}{dx^2}$ even though it is not quite a second derivative in the usual sense. The space $H^2_D(0, 2\pi)$ is a *Sobolev space.*

Proposition 4. *If $u \in H^2_D(0, 2\pi)$ then u is once continuously differentiable on $[0, 2\pi]$ (meaning it has a unique representative which is so differentiable) and has $u(0) = u(2\pi) = 0$.*

Proof. If one looks at the formula for Kf then it follows that when $f \in \mathcal{C}([02\pi])$,

$$\frac{d}{dx}Kf(x) = -\int_0^x f(s)ds + \frac{1}{2\pi}\int_0^{2\pi}\int_0^t f(s)dsdt. \tag{14}$$

This also extends by continuity to a map

$$K' : L^2(0, 2\pi) \longrightarrow \mathcal{C}([0, 2\pi]). \tag{15}$$

Thus, if $f \in L^2(0, 2\pi)$ then $\frac{d}{dx}Kf \in \mathcal{C}([0, 2\pi])$ and $u = Kf$ is therefore once differentiable. It also satisfies $u(0) = 0 = u(2\pi)$. $\square$

An operator such as D^2 is often thought of as an 'unbounded self-adjoint operator on $L^2(0, 2\pi)$ with domain $H^2_D(0, 2\pi) \subset L^2(0, 2\pi)$. In this case it is the inverse of $K : L^2(0, 2\pi) \longrightarrow H^2_D(0, 2\pi)$ which is a bounded, indeed compact, self-adjoint operator on $L^2(0, 2\pi)$.

What we will proceed to show is that if we take $V \in \mathcal{C}([0, 2\pi])$ a real-valued potential then the operator

$$D^2 + V : H^2_D(0, 2\pi) \longrightarrow L^2(0, 2\pi) \tag{16}$$

is similarly an unbounded self-adjoint operator.