BRIEF SOLUTIONS TO FINAL EXAM FOR 18.102, SPRING 2013

No books, papers, notes or electronic devices permitted. Attempt all 6 questions to obtain full marks.

All Hilbert spaces should be taken to be separable and non-trivial.

[The version you got in the exam did not have the non-triviality condition!]

Problem 1

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, χ_j of A_j, is integrable for each j. Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

Solution: The maximum and minimum of two real integrable functions is integrable, so

$$\chi_{A \cup B} = \max(\chi_A, \chi_B),\ \chi_{A \cap B} = \min(\chi_A, \chi_B)$$

are integrable if $\chi_A A$ and χ_B are integrable. Iterating this it follows that for any $R > 0$,

$$\chi_{B_N},\ B_N = [-R, R] \cap (\bigcup_{j=1}^{N} A_j)$$

is integrable. As $N \to \infty$ this sequence is monotonic increasing with integral bounded by $2N$ so by monotone convergence the limit, χ_R, exists, which shows that the characteristic function of $[-R, R] \cap A$ is in $\mathcal{L}^1$. It follows that the characteristic function of $\mathbb{R} \setminus A$ is locally integrable since $\chi_{\mathbb{R}\setminus A} = 1 - \chi_R$.

Problem 2

If H is a Hilbert space let $l^2(H)$ be the space of sequences $h : \mathbb{N} \longrightarrow H$ such that $\sum\limits_j \|u(j)\|_H^2 < \infty$. Show that this is a Hilbert space and that there is a bounded linear bijection $l^2(H) \longrightarrow H$ if and only if H is not finite dimensional.

Solution: If $\{e_i\}$ is an orthonormal basis of H – either finite or countable – then $u = \sum\limits_i (u, e_i)e_i$ is the corresponding convergent Fourier-Bessel series for any element, $u \in H$. If $\{u_j\}$ is an element of $l^2(H)$ it follows that

$$\|\{u_j\}\|^2 = \sum_{j=1}^{\infty} \sum_i |(u_j, e_i)|^2 < \infty.$$

Thus if $\phi : \mathbb{N} \times \{1, \dots, n\} \longrightarrow H$ or $\phi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ are bijections, depending on whether H as dimension $n > 0$ or ∞, then

$$\Phi : \{u_j\} \longrightarrow \{c_I\} \in l^2,\ c_I = (u_j, e_i) \text{ if } I = \phi(j, i) \tag{1}$$

is a linear bijection, with linearity following from the fact that (u_j, e_i) is linear in $\{u_j\}$. Since absolutely summable sequences can be reordered,

$$\|\{u_j\}\|^2 = \|\Phi(\{u_j\})\|_{l^2}^2$$

it follows that the left side defines a Hilbert norm on $l^2(H)$ which is complete and that as a Hilbert space, $l^2(H)$ is always isomorphic to l^2. If H is infinite dimensional then H is also isomorphic to l^2 and hence to $l^2(H)$. If H is finite dimensional then it cannot be isomorphic to the infinite dimensional space $l^2(H)$ since dimension is invariant under linear bijections.

Problem 3

Let A be a Hilbert-Schmidt operator on a separable Hilbert space H, which means that for some orthonormal basis $\{e_i\}$

$$\sum_i \|Ae_i\|^2 < \infty. \tag{2}$$

Using Bessel's identity to expand $\|Ae_i\|^2$ with respect to another orthonormal basis $\{f_j\}$ show that $\sum_j \|A^*f_j\|^2 = \sum_i \|Ae_i\|^2$. Conclude that the sum in (2) is independent of the othornormal basis used to define it and that the Hilbert-Schmidt operators form a Hilbert space.

Solution: Using Bessel's (or Parseval's) identity with respect to some orthonormal basis f_j,

$$\|Ae_i\|^2 = \sum_j |(Ae_i, f_j)|^2 = \sum_j |(e_i, A^*f_j)|^2$$

Since the absolutely summable series can be rearranged, applying the same identity for the first basis

$$\|A\|^2_{\mathrm{HS}} = \sum_i \|Ae_i\|^2 = \sum_{j,i} |(e_i, A^*f_j)|^2 = \sum_j \|A^*f_j\|^2.$$

Applying this identity again to any other basis shows that $\|A\|^2_{\mathrm{HS}}$ is independent of the basis. The Hilbert-Schmidt operators form a linear space since $\|cA\|^2_{\mathrm{HS}} = |c|^2\|A\|^2_{\mathrm{HS}}$ and $\|A+B\|^2_{\mathrm{HS}} \le 2\|A\|^2_{\mathrm{HS}} + 2\|B\|^2_{\mathrm{HS}}$. Choosing a basis with first element $v \in H$ of norm 1,

$$\|Av\|^2 \le \|A\|^2_{\mathrm{HS}} \Longrightarrow \|A\| \le \|A\|_{\mathrm{HS}}.$$

Consider the map

$$T : \mathrm{HS} \ni A \longrightarrow \{Ae_i\} \in l^2(H).$$

which is linear, since it is linear in each component. It is injective, since if A vanishes on a basis it vanishes on the closure of the span of the basis, i.e. on H. Moreover, T is surjective since if $\{v_i\} \in l^2(H)$ then

$$Au = \sum_i (u, e_i)v_i$$

converges in H – if H is infinite dimensional then

$$\|\sum_{i=m}^{n} (u, e_i)v_i\|^2 \le \sum_m^n |(u, e_i)|^2 \cdot \sum_m^n \|v_i\|^2.$$

and A is linear and bounded with $\|A\|^2_{\mathrm{HS}} = \|\{v_j\}\|^2_{l^2(H)}$. Thus the space of Hilbert-Schmidt operators is a Hilbert space since it is isometrically isomorphic to $l^2(H)$, shown to be a Hilbert space above.

Problem 4

Let B_n be a sequence of bounded linear operators on a Hilbert space H such that for each u and $v \in H$ the sequence $(B_n u, v)$ converges in $\mathbb{C}$. Show that there is a uniquely defined bounded operator B on H such that

$$(Bu, v) = \lim_{n\to\infty}(B_n u, v) \;\forall\; u, v \in H.$$

Solution: By assumption $B_n u$ converges weakly in H for each $u \in H$. So (by the uniform boundedness principle) is bounded with limit which can be denoted $Bu \in H$. By uniqueness of weak limits Bu depends linearly on u and by the uniform boundedness principle $\|B_n\|$ has an upper bound C and hence $\|Bu\| \le C\|u\|$ is a bounded operator. It is uniquely determined since the defining condition implies that $B_n u$ converges weakly to Bu for all u.

Problem 5

Suppose $P \subset H$ is a closed linear subspace of a Hilbert space, with $\pi_P : H \longrightarrow P$ the orthogonal projection onto P. If H is separable and A is a compact self-adjoint operator on H, show that there is a complete orthonormal basis of H each element of which satisfies $\pi_P A \pi_P e_i = t_i e_i$ for some $t_i \in \mathbb{R}$.

Solution: By definition the orthogonal projection onto P is the unique bounded self-adjoint operator π_P with $\pi_P^2 = \pi_P$ and range P. Thus $(\pi_P A \pi_P)^* = \pi_P A^* \pi_P$ is also self-adjoint and as the compact operators form an ideal is compact. Thus, by the Spectral Theorem, there exists an orthonormal basis of eigenfunctions of $\pi_P A \pi_P$ as desired, with all eigenvalues real.

Problem 6

Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, where $j = 1, 2, \dots,$ and $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. You may assume completeness in $L^2(\mathbb{R})$ and use the facts established in class that $-\frac{d^2 e_j}{dx^2} + x^2 e_j = (2j+1)e_j$, that $c_j = 2^{-j/2}(j!)^{-\frac{1}{2}}\pi^{-\frac{1}{4}}$ and that $e_j = p_j(x)e_0$ for a polynomial of degree j. Compute Ce_j and Ae_j in terms of the basis and hence arrive at a formula for de_j/dx. Use this to show that the sequence $j^{-\frac{1}{2}}\frac{de_j}{dx}$ is bounded in $L^2(\mathbb{R})$. Conclude that if

$$H^1_{\text{iso}} = \{u \in L^2(\mathbb{R}); \sum_{j\ge 1} j|(u, e_j)|^2 < \infty\} \tag{3}$$

then there is a uniquely defined operator $D : H^1_{\text{iso}} \longrightarrow L^2(\mathbb{R})$ such that $De_j = \frac{de_j}{dx}$ for each j.

Solution: By definition $H^1_{\text{iso}} \subset L^2(\mathbb{R})$ so if $u \in H^1_{\text{iso}}$ then

$$\|u\|^2_{\text{iso}} = \sum_{j\ge 1}(j+1)|(u, e_j)|^2 < \infty.$$

This is a Hilbert norm on H^1_{iso} since it is equivalent to the condition

$$Tu = \{(j+1)^{\frac{1}{2}}(u, e_j)\} \in l^2$$

which shows that H^1_{iso} is linear and then T is an isometric isomorphism to l^2 so H^1_{iso} is a Hilbert space.

Using the identities for the creation and annihilation operators,

$$Ce_j = (2j+1)^{\frac{1}{2}} e_{j+1},\ Ae_j = (2j)^{\frac{1}{2}} e_{j-1},\ j \ge 1,\ Ae_0 = 0.$$

Thus for the eigenfunctions, which are all in the Schwartz space,

$$\frac{d}{dx} e_j = \frac{1}{2}(A - C) = \frac{1}{2}(2j)^{\frac{1}{2}} e_{j-1} - \frac{1}{2}(2j+1)^{\frac{1}{2}} e_{j+1},\ e_{j-1} = 0.$$

So we may define

$$Du = \sum_{j \ge 0} (u, e_j) \left(\frac{1}{2}(2j)^{\frac{1}{2}} e_{j-1} - \frac{1}{2}(2j+1)^{\frac{1}{2}} e_{j+1} \right)$$

for $u \in H^1_{\text{iso}}$ since $\frac{1}{2}(2j)^{\frac{1}{2}}(u, e_j)$ and $\frac{1}{2}(2j+1)^{\frac{1}{2}}(u, e_j)$ are in l^2 with norms bounded by $\|u\|_{\text{iso}}$ and D is therefore a bounded linear operator, using the linearity of (u, e_j) and $\|D\| \le 2$.