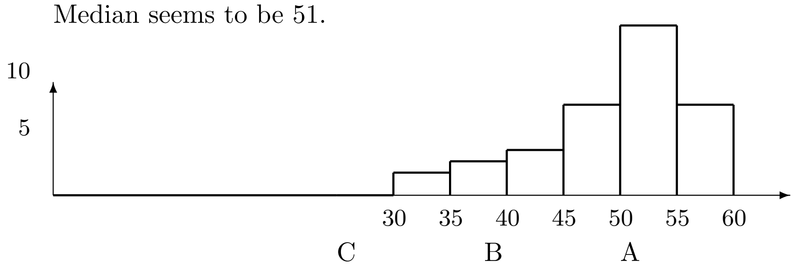
**SOLUTIONS FOR TEST 2 FOR SECTION 1 OF 18.100B/C WAS DUE 5PM TUESDAY NOVEMBER 23.**

Median seems to be 51.




(1) If $f : X \longrightarrow Y$ is a continuous map between metric spaces and $K \subset X$ is compact, prove that $f(K) \subset Y$ is compact using the definition of compactness through open covers.

This can be found in Rudin's book of course..

Solution: By definition a set in a metric space is compact if any open cover of it has a finite subcover. Let $U_\alpha$, $\alpha \in A$, be an open cover of $f(K)$. Since $f$ is continuous, the inverse image $f^{-1}(V)$ of any open set $V \subset Y$ is open in $X$. The open sets $f^{-1}(U_\alpha)$ form an open cover of $K$, since if $x \in K$ then $f(x) \in f(K)$ so $x \in U_\alpha$ for some $\alpha \in A$ and hence $x \in f^{-1}(U_\alpha)$. Since $K$ is compact this open cover has a finite subcover,

$$K \subset \bigcup_{i=1}^{N} f^{-1}(U_{\alpha_i}).$$

It follows that $f(K) \subset \bigcup_{i=1}^{N} U_{\alpha_i}$ since if $y \in f(K)$ then $y = f(x)$ for some $x \in K$ and hence $x \in f^{-1}(U_{\alpha_i})$ for some $i \leq N$ which implies $y = f(x) \in U_{\alpha_i}$. Thus any open cover of $f(K)$ has finite subcover so $f(K)$ is compact.

Comments: You could do the inclusions in a more sophisticated way, noting that $f^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha)$ and $f(f^{-1}(E)) \subset E$ (not equals in general). No need to prove that $f^{-1}(V)$ is open if $f$ is continuous and $V$ is open.

(2) Suppose $f : [0,1] \longrightarrow \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x - y|^{\frac{3}{2}} \ \forall\ x, y \in [0,1].$$

Explain why it follows that $f(0) = f(1)$.

Solution 1: Fix $x \in [0,1]$ then for $y \neq x$ the difference quotient satisfies

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq |x - y|^{\frac{1}{2}}.$$

The right side tends to zero as $y \to x$ (by the continuity of $|x|^{\frac{1}{2}}$, so

$$\lim_{y \to x} \frac{f(x) - f(y)}{x - y} = 0.$$

This is precisely the defining condition that $f$ is differentiable at $x$ and $f'(x) = 0$. Thus $f$ is differentiable on the whole interval and by the Mean Value Theorem there exists $c \in (0, 1)$ such that

$$f(1) - f(0) = f'(c)(1 - 0) = 0.$$

Solution 2: Divide the interval into $n$ equal parts and use the triangle inequality and the bound to see that

$$|f(1) - f(0)| \le \sum_{k=1}^{n} |f(\frac{k}{n}) - f(\frac{k-1}{n})| \le n \cdot n^{-\frac{3}{2}} = n^{-\frac{1}{2}}.$$

Now take the limit as $n \to \infty$ to see that $|f(1) - f(0)| = 0$ and hence $f(1) = f(0)$.

Comments: Differentiability is defined by Rudin even 'from the right and left' at the end-points. You could just note that continuity everywhere follows from the condition and with differentiability in the interior enough to apply the MVT. You could also just recall that Rudin proves that the vanishing of the derivative everywhere implies the constancy of the function. I was generally not impressed by arguments aiming to prove differentiability by contradiction.

(3) If $\alpha : [0, 1] \longrightarrow \mathbb{R}$ is given by

$$\alpha(x) = \begin{cases} x - 1 & 0 \le x < \frac{1}{2} \\ x + 1 & \frac{1}{2} \le x \le 1 \end{cases}$$

and $f = 2x$, explain why the Riemann-Stieltjes integral $\int_0^1 f d\alpha$ exists and compute its value, justifying your arguments carefully.

Solution: Since $\alpha$ is an increasing function and $f$ is continuous the existence of the Riemann-Stieltjes integral follows from a basic result to this effect in class or Rudin's book.

To compute the integral, note that $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 = x - 1$ and $\alpha_2 = 2I(x - \frac{1}{2})$ where

$$I(x - \frac{1}{2}) = \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x \ge \frac{1}{2} \end{cases}$$

is a step function. Both are increasing functions. From one of the basic properties of the integral from the book, $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$, which follows again from the continuity of $f$, imply that

$$\int_0^1 f d\alpha = \int_0^1 f d\alpha_1 + \int_0^1 f d\alpha_2.$$

Since addition of a constant to $\alpha$ does not change the definition of the integral, the first term here is a Riemann integral which can be evaluated by the fundamental theorem of calculus, since $2x = (x^2)'$

$$\int_0^1 f d\alpha_1 = \int_0^1 2x dx = (1)^2 - 0^2 = 1.$$

The integral when $\alpha$ is a step function is also evaluated in the book, as just being the product of the size of the jump and the value of the function, so

$$\int_0^1 f d\alpha_1 = 2(2 \times \frac{1}{2}) = 2$$

and finally

$$\int_0^1 f d\alpha = 3.$$

Alternatively, some people divided the integral up using a choice of $0 < \epsilon < \frac{1}{2}$ to see that

$$\int_0^1 f d\alpha = \int_0^{\frac{1}{2}-\epsilon} f d\alpha + \int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}} f d\alpha + \int_{\frac{1}{2}}^1 f d\alpha.$$

As before the first and the last integrals are Riemann integrals which can be evaluated using the FTC. The middle integral can be bounded by sup and inf of the integrand and Stieltjes length of the interval

$$(1-2\epsilon)(\frac{3}{2} - (-\frac{1}{2} - \epsilon)) \le \int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}} f d\alpha \le 1(\frac{3}{2} - (-\frac{1}{2} - \epsilon)).$$

This shows that as $\epsilon \to 0$ the second integral coverges to 2 and the other two terms coverge to 1/4 and 3/4 giving the same answer.

Comments: I did not expect you to prove the integrability of a continuous function. The major (and rather common) error was to ignore the jump in $\alpha$ by dividing the integral at $\frac{1}{2}$. The jump is still there, so the lower integral is NOT a Riemann integral – the function $\alpha$ is not differentiable on the closed interval. This probably goes back to a misunderstanding about the definition of differentiability on a closed interval – Rudin does demand differentiability at the end-points even though this is 'one-sided'.