# TEST 1 FOR SECTION 1 OF 18.100B/C

About grades. Each question was marked out of 20, so a total of 60 points. I gave less than 10 marks for a question if I did not see a method to solve it. Around about 13/14 marks means that there was clearly a method but with a major missing component (e.g. not using compactness in the last question, just assuming some sequence converges). Above 15 means that your solution was basically correct but that there is some logical flaw – typically that you said something that is not true. So, to get 20 marks for a question you have to be completely right. As a bonus for reading these comments I will let you know that one of the questions below will be on the final!

I had hoped that people would do better on this, but I can see that maybe one of the questions should have been easier and that it was a bit relentless. It doesn't really make sense to assign letter grades based on this test, but here is a rough idea – based unrealistically on this test alone.
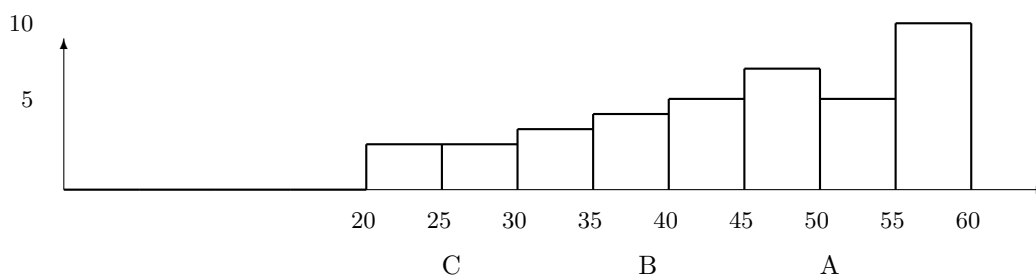
(1) Suppose $E \subset \mathbb{R}$ has the property that for every non-empty $B \subset E$ which is bounded, $\sup B$ and $\inf B$ are in E. Show that E is closed with respect to the standard metric.

Solution: By a theorem in Rudin, any limit point of a set in a metric space is the limit of a sequence, the sequence in the set, E, the limit in the metric space. Thus if $x \in E'$ is a limit point of E then there is a sequence $x_n \in E$ with $x_n \to x$ in $\mathbb{R}$. Consider all $n \in \mathbb{N}$ such that $x_n \leq x$. If this is infinite, then there is a subsequence x_{n_j} with $x_{n_j} \leq x$. If not then there is a subsequence x_{n_j} with $x_{n_j} > x$. In either case, $x_{n_j} \to x$ so we can change notation and just assume that either $x_n \leq x$ or $x_n > x$ for all n. Let $B \subset E$ be the range of this sequence, this set is bounded, since any convergent sequence is bounded. Moreover in the first case $\sup B = x$ and in the second $\inf B = x$, since other wise the sequence could not converge to x. Thus $x \in E$ and hence $E' \subset E$ and E is necessarily closed.

Of course there are many variants of this. One can certainly avoid using sequences. For instance, suppose $x \in E'$ but $x \notin E$. Then the sets $B(x, 1/n) \cap E$ are all infinite, for $n \in \mathbb{N}$. Consider $(x, 1/n) \cap E$; either this is infinite for all n or else it is empty for large n, in which case $(x - 1/n, x) \cap E$ must be infinite for all n. So, we can choose either $x_n \in (x, x + 1/n)$ for all n or $x_n \in (x - 1/n, x)$ for all n. Let B be the subset of E consisting of these

choices, then $x = \inf B$ in the first case and $x = \sup B$ in the second case, and in both cases B is bounded. Thus in fact $x \in E$ by the assumption, contradicting the assumption that $x \notin E$. Thus E is closed.

I rather like the following proof from Yunjian Xu which neatly avoids the division into two pieces: Let p be a limit point of E. Then for every $n \in \mathbb{N}$, $D_n = B(p, 1/n) \cap E$ is a bounded, infinite subset of E, so by assumption $q_n = \sup D_n \in E$. This sequence is bounded since it lies in $B(p, 1)$; let B be its range. This is again a bounded nonempty subset of E and we claim $p = \inf B$, so $p \in E$. Indeed q_n is a non-increasing (the sets are getting smaller) sequence which is bounded below so it converges to the infimum of its range, but since $|q_n - p| \leq 1/n$ the limit must be p.

Main shortcomings: Not making sure that x was the sup or inf of a chosen subset. Minor problems included assuming that just because $(x-1, x) \cap E$ was infinite then x has to be the supremum – $E \cap (x-1, x) \subset (x-1, x-\frac{1}{2})$ is a possibility (but then of course $x = \inf(x, x+1) \cap E$).

(2) Show that in any metric space, the closure of a connected set is connected.

Solution: Let $E \subset X$ be a connected subset of the metric space X. Consider $\overline{E} = E \cup E'$, the closure of E (using results from Rudin). Now, suppose that $\overline{E} = C \cup D$ where $C \cap D = \emptyset$, neither C nor D is empty and $\overline{C} \cap D = C \cap \overline{D} = \emptyset$. Then $E = A \cup B$ where $A = C \cap E$ and $B = D \cap E$. First consider the possibility that $A = \emptyset$. This implies that $C \subset E'$ so $E \subset D$ which implies that $\overline{D} = \overline{E}$ and hence that $C = \emptyset$, contradicting the assumption that it is not. The same argument applies to show $B \neq \emptyset$ (by symmetry between the two sets). Thus, A and B must both be non-empty. Hence, by the connectedness of E, $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$. Since $\overline{B} \subset \overline{D}$ whenever $B \subset D$ these in turn imply that either $\overline{C} \cap D \neq \emptyset$ or $C \cap \overline{D} \neq \emptyset$ contradicting the properties of C and D. Thus, no such sets can exist and hence that $\overline{E}$.

I like the following proof from Ammar Ammar's answer. Suppose E is connected, consider a decomposition $\overline{E} = A \cup B$ where A and B are separated, i.e. $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Since $\overline{E}$ is closed, taking closures, $\overline{E} = \overline{A} \cup \overline{B} = A \cup B$, and as they are separated, $\overline{A} \cap B = \emptyset$ implies that $\overline{A} \subset A$, so A is closed and similarly B is closed. Thus, $E = (A \cap E) \cup (B \cap E)$ is a decomposition into separated sets, hence either $A \cap E = \emptyset$ which implies $E \subset B \cap E$ and hence $\overline{E} \subset B$ and so $A = \emptyset$ or $B \cap E = \emptyset$ which implies $E \subset A$ and hence $\overline{E} \subset A$ and $B = \emptyset$. Thus $\overline{E}$ is connected.

(3) Let K be a non-empty compact set in a metric space X and suppose $p \in X \setminus K$. Show that there exists a point $q \in K$ such that

$$d(p, q) = \inf\{d(p, x); x \in K\}.$$

Solution: The set $\{d(p, x); x \in K\}$ is a non-empty subset of $(0, \infty)$ so the infimum exists and there is a sequence $x_n \in K$ such that $d(p, x_n) \to D = \inf\{d(p, x); x \in K\}$. Now, as a sequence in a compact set, $\{x_n\}$ has a convergent subsequence. Since $d(p, x_{n_k})$ also converges to D, we may just assume that $x_n \to q$ in X. Since compact sets are closed, $q \in K$ and we just need to check that $d(p, q) = D$. By the definition of infimum and the convergence of the distance, give $\epsilon > 0$ there exists n such that

$|d(x, x_n) - D| < \epsilon/2$ and $d(x_n, q) < \epsilon/2$ but this implies that

$$|d(p, q) - D| < |d(p, q) - d(p, x_n)| + |d(p, x_n) - D| < \epsilon$$

for any $\epsilon > 0$. Thus $d(p, q) = D$ as desired and the infimum of the distance is attained.

Here is a direct approach that a couple of people used. Set $D = \inf\{d(p, x); x \in K\}$ and suppose that this is *not* attained on K, so for all $x \in K$, $d(p, x) > D$. Thus

$$K \subset \bigcup_{x \in K} B(x, \frac{1}{2}(d(p, x) - D))$$

is an open cover, which therefore has a finite subcover since K is compact. Let x_i, $i = 1, \ldots, N$ be the centers of such a cover with $\epsilon_i = \frac{1}{2}(d(p, x_i) - D)$ and $\epsilon = \min_i \epsilon_i > 0$. Then, each $x \in K$ is in one of these balls, so from the triangle inequality, for the appropriate i,

$$d(p, x) \geq d(p, x_i) - \epsilon_i \geq D + \epsilon_i \geq D + \epsilon.$$

This however shows that D is not the infimum as it is defined to be, so there must be a point $q \in K$ with $d(p, q) = D$.

There is an even simpler direct approach used by several people. Suppose that $d(p, x) > r = \inf\{d(y, p); y \in K\}$ for all $x \in K$. Then the open sets

$$G(n) = \{x \in K; d(x, p) > r + 1/n\}$$

form an open cover of K which therefore must have a finite subcover by compactness. Since the $G(n)$ increase with n, $K \subset G(N)$ for some N and hence $d(x, p) > r + 1/N$ for all $x \in K$, contradicting the definition of the infimum.

Another variant of this is to define $r = \inf\{d(x, p); x \in K\}$ and then to set

$$K(n) = K \cap \{x \in X; d(x, p) \leq r + 1/n\}.$$

Since the second sets are closed, these are compact sets, being closed subsets of K, which are non-empty, by the definition of infimum, and decreasing as n increases. Thus, by a theorem in Rudin, $T = \cap_n K(n) \neq \emptyset$. If $q \in T \subset K$ then $d(p, q) = r$ since $d(p, q) \leq r + 1/n$ for all n and $d(p, q) \geq r$.