

TEST 1 FOR SECTION 1 OF 18.100B/C
THURSDAY 14 OCTOBER 2010 IN CLASS 9:30-11

These are practice questions – from which the test will be selected.

In the test itself you should give full, clear answers – you can use results from Rudin (or the lectures but really they are all in Rudin) if you can quote them directly. You are not supposed to use results from the homework in this way.

- (1) Suppose $E \subset \mathbb{R}$ has the property that for every $B \subset E$ which is bounded, $\sup B$ and $\inf B$ are in E . Show that E is closed with respect to the standard metric.
- (2) Which of the following metric spaces are compact? Justify your answers: Either show that the set is compact or, if it is not, give an example of an open cover having no finite subcover.
 - a) X is a countable (infinite) set and $d(x, y) = 1$ for any two distinct points $x, y \in X$.
 - b) $X = \{(x, y) \in \mathbb{R}^2; 0 \leq x \leq y \leq 1\}$ with Euclidean distance.
 - c) $X = \{(x, y) \in \mathbb{R}^2; 0 \leq x < y \leq 1\}$ with Euclidean distance
- (3) Show that in any metric space, the closure of a connected set is connected.
- (4)
 - (a) Show that any closed subset of a complete metric space is complete.
 - (b) Conversely, show that if (X, d) is a metric space, $E \subset X$ and (E, d) is a complete metric space then E is a closed subset of X .
- (5) Suppose that $\{p_n\}$ is a sequence in a metric space, X , and $p \in X$. Assuming that every subsequence of $\{p_n\}$ itself has a subsequence which converges to p show that $p_n \rightarrow p$.
- (6) Give a counterexample to each of the following statements:
 - (a) Subsets of \mathbb{R} are either open or closed
 - (b) A closed and bounded subset of a metric space is compact.
 - (c) In any metric space the complement of a connected set is connected.
 - (d) The closure of the an open ball $\{x \in X; d(p, x) < r\}$ is the closed ball $\{x \in X; d(p, x) \leq r\}$.
- (7) Let $K_i, i = 1, \dots, N$, be a finite number of compact sets in a metric space X . Show that $\bigcup_{i=1}^N K_i$ is compact.
- (8) Let K be a non-empty compact set in a metric space X and suppose $p \in X \setminus K$. Show that there exists a point $q \in K$ such that
$$d(p, q) = \inf\{d(p, x); x \in K\}.$$
- (9) Let $x_n, n = 1, 2, \dots$, be a sequence of real numbers with $x_n \rightarrow 0$. Show that there is a subsequence $x_{n(k)}, k = 1, 2, \dots$, such that $\sum_k |x_{n(k)}| < \infty$.
- (10) Give examples of:
 - (a) A countable subset of \mathbb{R}^2 which is infinite and closed.
 - (b) A subset of the real interval $[-2, 2]$ which contains $[0, 1]$ but is not compact.
 - (c) A metric space in which all subsets are compact.
 - (d) A cover of $(0, 1]$ as a subset of \mathbb{R} which has no finite subcover.

- (11) a) Let X be a non-empty metric space with metric d and $p \in X$ given. Let $q \notin X$ be an external point. Show that there is a unique metric d_Y on $Y = X \cup \{q\}$ satisfying

$$d_Y(x, x') = d(x, x'), \quad \forall x, x' \in X,$$

$$d_Y(q, x) = d(p, x) + 1, \quad \forall x \in X.$$

- b) Show that with this metric Y is not connected.
- (12) Let X be a metric space and suppose $A \subset X$. Let A° be the union of all those open sets in X which are subsets of A . Show that the complement of A° is the closure of the complement of A .
- (13) Let $\{x_n\}$ be a sequence of positive real numbers such that $x_n^3 > n$. Show that $1/x_n \rightarrow 0$ as $n \rightarrow \infty$ with respect to the usual metric on \mathbb{R} .