HOMEWORK 9 FOR 18.100B/C, FALL 2010
DUE THURSDAY 18 NOVEMBER

As usual, homework is due in 2-108 by 11AM on Thursday 18 November, or by email before 5PM on the same day.

(1) A function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be 'Lipschitz continuous' (or just 'Lipschitz') if there exists a constant A such that
$$|f(x) - f(y)| \le A|x - y| \ \forall \ x, y \in [a, b].$$
Show that if $f : [a, b] \longrightarrow \mathbb{R}$ is differentiable and $f' : [a, b] \longrightarrow \mathbb{R}$ is bounded then f is Lipschitz.

(2) Suppose that $g : [0, 1] \longrightarrow \mathbb{R}$ is a Lipschitz function and that $f : [0, 1] \longrightarrow [0, 1]$ is a differentiable functions satisfying
$$f'(x) = g(f(x)) \ \forall \ x \in [0, 1].$$
Show that $f' : [0, 1] \longrightarrow \mathbb{R}$ is Lipschitz.

(3) Rudin Chap 7, No 1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

(4) Rudin Chap 7, No 2. If $\{f_n\}$ and $\{g_n\}$ are sequences of functions defined on a set E and each converges uniformly on E prove that $\{f_n + g_n\}$ converges uniformly on E. If in addition both sequences consist of bounded functions, prove that $f_n g_n$ converges uniformly on E.

(5) Rudin Chap 7, No 6. Prove that the series of functions
$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$
on the real line converges uniformly on any bounded subset of the reals but does not converge absolutely for any value of x.