HOMEWORK 9 FOR 18.100B/C, FALL 2010
WAS DUE THURSDAY 18 NOVEMBER

(1) A function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be 'Lipschitz continuous' (or just 'Lipschitz') if there exists a constant A such that
$$|f(x) - f(y)| \leq A|x - y| \ \forall \ x, y \in [a, b].$$
Show that if $f : [a, b] \longrightarrow \mathbb{R}$ is differentiable and $f' : [a, b] \longrightarrow \mathbb{R}$ is bounded then f is Lipschitz.

Solution. Set $A = \sup_{x \in (a,b)} |f'(x)|$ which is finite by hypothesis. By the Mean Value Theorem, if $x < y$ there exists some $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$. Thus
$$|f(x) - f(y)| \leq A|x - y|.$$
This certainly holds if $x = y$ and follows by symmetry if $x > y$ so always holds and f is therefore Lipschitz. $\qquad\square$

(2) Suppose that $g : [0, 1] \longrightarrow \mathbb{R}$ is a Lipschitz function and that $f : [0, 1] \longrightarrow [0, 1]$ is a differentiable functions satisfying
$$f'(x) = g(f(x)) \ \forall \ x \in [0, 1].$$
Show that $f' : [0, 1] \longrightarrow \mathbb{R}$ is Lipschitz.

Solution. Since a differentiable function is continuous, f is continuous on $[0, 1]$. Since g, being Lipschitz, is continuous and defined on $[0, 1]$ it follows from a theorem in Rudin that the composite, $g \circ f$, is continuous on $[0, 1]$ and hence is bounded. Thus, from the differential equation, $|f'| \leq M$ is also bounded and hence, using the preceding problem, f is Lipschitz. So, we need to show that the composte $f' = g \circ f$ is also Lipschitz. This follows by applying the two Lipschitz inequalities:-
$$|g \circ f(x) - g \circ f(y)| = |g(f(x)) - g(f(y))| \leq A|f(x) - f(y)| \leq AM|x - y|$$
where A is the Lipschitz constant for g. $\qquad\square$

(3) Rudin Chap 7, No 1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Solution. If f_n is a uniformly convergent sequence of functions on a set E then, applying the definition of uniform convergence, there exists N such that if $n \geq N$,
$$|f_n(e) - f_N(e)| \leq 1 \implies |f_n(e)| \leq 1 + \sup_E |f_N|.$$
Thus the boundedness of $f_1, \ldots, f_N$ implies the uniform boundedness of all the functions in the sequence. $\qquad\square$

(4) Rudin Chap 7, No 2. If $\{f_n\}$ and $\{g_n\}$ are sequences of functions defined on a set E and each converges uniformly on E prove that $\{f_n + g_n\}$ converges uniformly on E. If in addition both sequences consist of bounded functions, prove that $f_n g_n$ converges uniformly on E.

Solution. Directly from the definition of uniform convergence, given $\epsilon > 0$ there exist N_1 and N_2 such that $n > N_1$ implies $|f_n(e) - f(e)| < \epsilon/2$ and $n > N_2$ implies $|g_n(e) - g(e)| < \epsilon/2$ where f and g are the limiting functions. Thus if $n > N_1 + N_2$ then $|(f_n + g_n)(e) - (f + g)(e)| < \epsilon$ so the sum gives a uniformly convergent sequence.

For the product we assume that both f_n and g_n are bounded, so by the preceding result, both sequences are uniformly bounded. If M is an upper bound for all $|f_n|$ and $g_n|$ it follows that

$$|f_n g_n(e) - fg(e)| \leq |f_n(e) - f(e)|g_n(e)| + |f(e)||g_n(e) - g(e)| < M\epsilon$$

whenever $n > N_1 + N_2$. Thus $f_n g_n \to fg$ uniformly on E. $\qquad\square$

(5) Rudin Chap 7, No 6. Prove that the series of functions

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

on the real line converges uniformly on any bounded subset of the reals but does not converge absolutely for any value of x.

Solution. The series $\sum_n \frac{(-1)^n}{n}$ and $\sum_n \frac{1}{n^2}$ are both convergent, with the first not being absolutely convergent. Now

$$\sum_{n=1}^{m} (-1)^n \frac{x^2 + n}{n^2} = x^2 \sum_{n=1}^{m} \frac{(-1)^n}{n^2} + \sum_{n=1}^{m} \frac{(-1)^n}{n^2}.$$

So, for x in any bounded subset of $\mathbb{R}$, the first series on the right is uniformly and absolutely Cauchy hence uniformly and absolutely convergent. The second series is convergent but not absolutely convergent. So the given series is uniformly convergent on any bounded set but cannot be absolutely convergent for any single value of x since this would imply the absolute convergence of $\sum_{n=1}^{\infty} (-1)^n/n$. $\qquad\square$