SOLUTIONS TO HOMEWORK 8 FOR 18.100B/C, FALL 2010
WAS DUE FRIDAY 12 NOVEMBER

(1) Rudin Chap 6 No 2. Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and non-negative, $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x)dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$.

Solution. If f is not identically zero then there exists $c \in [a, b]$ such that $|f(c)| > 0$. Since f is assumed to be continuous, there exists $\delta > 0$ such that $|x-c| \leq \delta$ implies $|f(x)-f(c)| < |f(c)|/2$ and hence that $|f(x)| > |f(c)|/2$. Since the length of this interval is at least δ (it might be close to one of the ends) it follows that

$$\int_a^b |f(x)|dx \geq \frac{1}{2}|f(c)|\delta > 0.$$

Thus, if $\int_s^b |f(x)|dx = 0$ and f is continuous it follows that $f \equiv 0$ on $[a, b]$. □

(2) Rudin Chap 6 No 4. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = 0$ when x is irrational and $f(x) = 1$ when x is rational. Show that $f \notin \mathcal{R}$ (the space of Riemann integrable functions) on any interval $[a, b]$ where $a < b$.

Solution. Since both the rationals and the irrationals are dense it follows that for any interval $[x_{n-1}, x_n]$, with $x_n > x_{n-1}$, $\inf f = 0$ and $\sup f = 1$. Thus for any partition of an interval of non-zero length, l, where we can always drop the segments of zero length, $L(\mathcal{P}, f) = 0$, $U(\mathcal{P}, f) = l$. Thus these are equal to the lower and upper integrals, which are therefore not equal, so f is not Riemann integrable on any non-empty interval. □

Note that this is an example of a function which is Lebesgue by not Riemann integrable, not that we can prove this very easily, but only because we have not defined Lebesgue integrability.

(3) Rudin Chap 6 No 5. Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and that $f^2 \in \mathcal{R}$ does it follow that $f \in \mathcal{R}$? What if $f^3 \in \mathcal{R}$?

Solution. In general it does not follow from the Riemann integrability of f^2 that a bounded function is Riemann integrable. An example is given by $g = f - \frac{1}{2}$ where f is the function in the preceding problem. Then $g(x) = \pm\frac{1}{2}$ as x is irrational or rational so g^2 is constant, hence Riemann integrable, but g is not – since if it was it would follows that f was also Riemann integrable. □

(4) Rudin Chap 6 No 8. Suppose that $f : [1, \infty) \longrightarrow \mathbb{R}$ is non-negative and monotonic decreasing. Show that $\lim_{b\to\infty} \int_1^b f(x)dx$ exists (and is finite) if and only if $\sum\limits_{n=1}^{\infty} f(n)$ exists (and is finite).

Solution. If f is non-negative and monotonic decreasing then, by a theorem in Rudin, it is integrable on any finite interval. Thus $F(b) = \int_1^b f(x)dx$ does exist for $b \geq 1$. Moreover, since f is non-negative, it is an increasing function of b, $F(b') - F(b) = \int_b^{b'} f dx \geq 0$ for $b' > b$. Thus the limit exists if and only if F is bounded above. Since f is monotonic decreasing, it follows that $x \in [n, n+1]$ implies $f(n) \geq f(x) \geq f(n+1)$. Thus,

$$F(x) = \sum_{i=1}^{n-1} \int_i^{i+1} + \int_n^x > \sum_{i=1}^{n-1}$$
$$\Longrightarrow F(x) ge \sum_{i=1}^{n-1} f(i)$$
$$\text{and } f(x) \leq \sum_{i=1}^{n-1} f(i+1).F(x)$$

It follows that the sequence of partial sums of the series $\sum_{n=1}^{\infty} f(n)$ is bounded if and only if F is bounded, hence the limits either both exist or both are infinite. □

(5) Suppose that $\alpha : [a, b] \longrightarrow \mathbb{R}$ is monotonic increasing and $f \in \mathcal{R}(\alpha)$ is real-valued and Riemann-Stieltjes integrable on $[a, b]$. Show that for every $\epsilon > 0$ there is a continuous function $g : [a, b] \longrightarrow \mathbb{R}$ such that $\int_a^b |f - g| d\alpha < \epsilon$. For a hint, see Rudin Chap 6 No 12.

Solution. If $\mathcal{P}$ is a partition of $[a, b]$ the function associated by Rudin to the partition and f is defined by

$$g(x) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

on $[x_{i-1}, x_i]$ is linear on each segment and takes the value $f(x_{i-1})$ at $t = x_{i-1}$ and $f(x_i)$ at $t = x_i$. It is in fact given by 'linear interpolation' between these two values. It follows that g is continuous, since it is continuous on each open interval and the limits from above and below at the ends exist and are equal.

On the other hand, given $\epsilon > 0$ we may choose $\mathcal{P}$ so that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \epsilon$$

by the assumed integrability of f. For $x \in [x_{i-1}, x_i]$ $f(x) - g(x) \leq M_i f - m_i f$ and similarly $g(x) - f(x) \leq M_i - m_i$ where $M_i = \sup_{[x_{i-1}, x_i]} f$, $m_i = \inf_{[x_{i-1}, x_i]} f$, so $|f(x) - g(x)| \leq M_i - m_i$ and it follows that

(1) $$0 \leq U(\mathcal{P}, |f - g|, \alpha) < \epsilon.$$

It follows that $\int_a^b |f - g| d\alpha < \epsilon$. Thus for any $\epsilon > 0$ there is always a continuous function g (depending of course on ϵ in general) such that $\int_a^b |f - g| d\alpha < \epsilon$. □