HOMEWORK 7 FOR 18.100B/C, FALL 2010
DUE THURSDAY 4 NOVEMBER

As usual due in 2-108 or lecture by 11AM or by email before 5PM.

(1) Let $K_1,\ K_2 \subset M$ be two compact subsets of a metric space (M, d). Show that there exist points $p \in K_1$ and $q \in K_2$ such that
$$d(p, q) = \sup_{y \in K_2} \inf_{x \in K_1} d(x, y).$$
Define
$$D(K_1, K_2) = \max\left(\sup_{y \in K_2} \inf_{x \in K_1} d(x, y), \sup_{x \in K_1} \inf_{y \in K_2} d(x, y)\right).$$
Show that D defines a metric on the collection of (non-empty) compact subsets of M.

(2) If $f : [a, b] \longrightarrow \mathbb{R}$ is differentiable (where $a < b$) and $f'(x) \neq 0$ for all $x \in (a, b)$ show that $f(b) \neq f(a)$.

(3) Rudin Chap 5 No 4. If C_i for $0 \leq i \leq n$ are real constants such that
$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$
show that the equation
$$C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n = 0$$
has at least one real solution x in the interval $(0, 1)$.

(4) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable and that $f'(x) \neq 1$ for all $x \in \mathbb{R}$ show that there can be *at most one* $x \in \mathbb{R}$ such that $f(x) = x$ ('a fixed point of f').

(5) Rudin Chap 5 No 15. Suppose $a \in \mathbb{R}$, f is a twice-differentiable real function on (a, ∞) and M_0, M_1 and M_2 are the suprema of $|f(x)|$, $|f'(x)|$ and $|f''(x)|$ on (a, ∞) (so all are assumed to be finite). Prove that
$$M_1^2 \leq 4M_0 M_2.$$
[There is a hint in Rudin, namely Taylor's theorem shows that given any $h > 0$ and $x \in (a, \infty)$ there is $\xi \in (x, x + 2h)$ such that
$$f'(x) = \frac{1}{2h}(f(x + 2h) - f(x)) - hf''(\xi).$$
Use this to show that $|f'(x)| \leq hM_2 + \frac{M_0}{h}$. For what value of h is the RHS smallest?]