HOMEWORK 7 FOR 18.100B/C, FALL 2010
DUE THURSDAY 4 NOVEMBER

As usual due in 2-108 or lecture by 11AM or by email before 5PM.

(1) Let K_1, $K_2 \subset M$ be two compact subsets of a metric space (M, d). Show that there exist points $p \in K_1$ and $q \in K_2$ such that

$$d(p, q) = \sup_{y \in K_2} \inf_{x \in K_1} d(x, y).$$

Define

$$D(K_1, K_2) = \max\left(\sup_{y \in K_2} \inf_{x \in K_1} d(x, y), \sup_{x \in K_1} \inf_{y \in K_2} d(x, y)\right).$$

Show that D defines a metric on the collection of (non-empty) compact subsets of M.

Solution(s): For each $y \in K_2$ we know that there exists a point $x' \in K_1$ such that $d(y, x') = \inf_{x \in K_1} d(x, y)$. This was in an earlier homework, using sequences, and we now know that it follows from the fact that $d(y, \cdot)$ is a continuous function of the second variable and hence attains its infimum on any compact set. Now, from the definition of $L = \sup_{y \in K_2} \inf_{x \in K_1} d(x, y)$ there must exist a sequence of pairs $(y_n, x'_n) \in K_2 \times K_1$ such that $d(y_n, x'_n) \to L$. Since K_1 and K_2 are compact we can pass to subsequences, first so that $x'_n \to q$ and then so that $y_n \to p$ and it follows that $d(p, q) = L$ since

$$\begin{aligned} |L - d(y', x')| &\le |L - d(y_n, x'_n)| + |d(y_n, x') - d(y_n, x'_n) + |d(y_n, x') - d(y', x')| \\ &\le |L - d(y_n, x'_n)| + d(x', x'_n) + |d(y_n, y')| \to 0. \end{aligned}$$

Of course, $p \in K_2$ and $q \in K_1$ since these sets are closed, hence compact.

Perhaps a better way to see this first part is to see check that the function $f(y) = \inf_{x \in K_1} d(x, y)$ defined on K_2 is continuous, hence it attains its supremum and this gives a pair (p, q). Continuity follows from the fact that, for fixed $y' \in K_2$ if $y \in B(y', \epsilon) \cap K_2$ then $|d(x, y) - d(x, y')| < \epsilon$. Thus the infimum, $f(y) < f(y') + \epsilon$ since there exists $x' \in K_1$ such that $d(x', y') = f(y')$ and hence $d(x, y) < f(y') + \epsilon$. Moreover, $y' \in B(y, \epsilon)$ and there exists $x \in K_1$ such that $d(x, y) = f(y)$, $f(y') < f(y) + \epsilon$. Thus $|f(y) - f(y')| < \epsilon$ and f is continuous.

Now, to see that D as defined is a metric, first note that it is non-negative and by definition symmetric, $D(K_1, K_2) = D(K_2, K_1)$. Since the infimum is always zero, $D(K, K) = 0$. To see that $D(K_1, K_2) \neq 0$ when $K_1 \neq K_2$ are both non-empty observe that, after exchanging the labels if necessary, there is a point $y \in K_2 \setminus K_1$. Then $\inf_{x \in K_1} d(x, y) > 0$ since it is realized at a point $x \in K_1$ and necessarily $x \neq y$, so $D(K_1, K_2) > 0$.

So, only the triangle inequality remains. Let me do this somewhat geometrically. Consider an arbitrary point $p \in K_1$ and select $x \in K_2$ such that $d(p, x) = \inf_{x' \in K_2} d(p, x')$. Having chosen this point, choose $y \in K_3$ such that $d(x,y) = \inf_{y' \in K_3} d(x,y')$. The triangle inequality gives

$$d(p,y) \le d(p,x) + d(x,y) \le D(K_1,K_2) + D(K_2,K_3)$$

since both terms on the right are infimums. Now, since $y \in K_3$,

$$\inf_{z \in K_3} d(p,x) \le d(p,y) \le D(K_1,K_2) + D(K_2,K_3).$$

Taking the supremum over $p \in K_1$ then gives

$$\sup_{p \in K_1} \inf_{z \in K_3} d(p,x) \le D(K_1,K_2) + D(K_2,K_3).$$

Reversing the roles of K_3 and K_1 then shows that

$$D(K_1,K_3) \le D(K_1,K_2) + D(K_2,K_3).$$

One does not need to go through the 'geometrical picture' if you are confident (competent?) with your infs and sups. Take the triangle inequality for an arbitrary triple, $p_i \in K_i$:

$$d(p_1,p_3) \le d(p_1,p_2) + d(p_2,p_3).$$

Now take the infimum of both sides over $p_3 \in K_3$ and use the definition of D to see that

$$\inf_{p_3 \in K_3} d(p_1,p_3) \le d(p_1,p_2) + \inf_{p_3 \in K_3} d(p_2,p_3) \le d(p_1,p_2) + D(K_2,K_3).$$

Now we may take the infimum over $p_2 \in K_2$ since it only appears on the right to get

$$\inf_{p_3 \in K_3} d(p_1,p_3) \le \inf_{p_2 \in K_2} d(p_1,p_2) + D(K_2,K_3) \le D(K_1,K_2) + D(K_2,K_3)$$

and then proceed as before – take the sup over p_1.

(2) If $f : [a,b] \longrightarrow \mathbb{R}$ is differentiable (where $a < b$) and $f'(x) \neq 0$ for all $x \in (a,b)$ show that $f(b) \neq f(a)$.

Solution. Since f is differentiable on $[a,b]$ it is continuous on $[a,b]$ by a Theorem in Rudin. The mean value theorem then shows that there exists $x \in (a,b)$ such that $f(b) - f(a) = f'(x)(b-a)$. Thus if $f'(x) \neq 0$ for all $x \in (a,b)$ then $f(b) \neq f(a)$.

(3) Rudin Chap 5 No 4. If C_i for $0 \le i \le n$ are real constants such that

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

show that the equation

$$C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n = 0$$

has at least one real solution x in the interval $(0,1)$.

Solution. The crucial obsevation is that if

$$p(x) = C_0 x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \cdots + C_n \frac{x^{n+1}}{n+1} \text{ then}$$
$$p'(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n.$$

Certainly $p(0) = 0$ and by assumption, $p(1) = 0$. As a polynomial, $p : [0,1] \longrightarrow \mathbb{R}$ is (infinitely) differential, so by the mean value theorem there is a point $x \in (0,1)$ such that

$$0 = p(1) - p(0) = p'(x)(1-0) = p'(x)$$

as desired.

(4) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable and that $f'(x) \neq 1$ for all $x \in \mathbb{R}$ show that there can be *at most one* $x \in \mathbb{R}$ such that $f(x) = x$ ('a fixed point of f').

Solution. Suppose there were two distinct points $x_1 < x_2$ with $f(x_1) = x_1$ and $f(x_2) = x_2$. Since f is differentiable on the real line, the function $g(x) = f(x) - x$ is continuous on $[x_1, x_2]$ and differentiable so by the mean value theorem there exists $x \in (x_1, x_2)$ such that $0 = g(x_2) - g(x_1) = g'(x)(x_2 - x_1) = (f'(x) - 1)(x_2 - x_1)$. By assumption, $f'(x) \neq 1$, so this is a contradiction and there can be at most one fixed point of f.

(5) Rudin Chap 5 No 15. Suppose $a \in \mathbb{R}$, f is a twice-differentiable real function on (a, ∞) and M_0, M_1 and M_2 are the suprema of $|f(x)|$, $|f'(x)|$ and $|f''(x)|$ on (a, ∞) (so all are assumed to be finite). Prove that

$$M_1^2 \leq 4M_0M_2.$$

[There is a hint in Rudin, namely Taylor's theorem shows that given any $h > 0$ and $x \in (a, \infty)$ there is $\xi \in (x, x + 2h)$ such that

$$f'(x) = \frac{1}{2h}(f(x + 2h) - f(x)) - hf''(\xi).$$

Use this to show that $|f'(x)| \leq hM_2 + \frac{M_0}{h}$. For what value of h is the RHS smallest?]

Solution. As Rudin suggests, using the twice-differentiability of f, which means that f' is differentiable, apply Taylor's theorem with remainder term to see that for any $h > 0$ there exists $\xi \in (x, x + 2h)$ such that

$$f(x+2h) = f(x)+2hf'(x)+\frac{h^2}{2}f''(\xi) \Longrightarrow f'(x) = \frac{1}{2h}(f(x+2h)-f(x))-hf''(\xi).$$

Now use the definitions of M_0 and M_2 to see that

$$|f'(x)| \leq= \frac{M_0}{h} + hM_2 \forall h > 0.$$

If $M_2 = 0$ then letting $h \to \infty$ shows that $f'(x) = 0$ and similarly if $M_0 = 0$ then letting $h \to 0$ leads to the same conclusion. So we may set $h = M_0^{\frac{1}{2}} M_2^{-\frac{1}{2}}$ and deduce that

$$|f'(x)| \leq 2(M_0M_2)^{\frac{1}{2}}. \tag{1}$$

Taking the supremum over $x \in (a, \infty)$ and squaring gives the desired estimate.