HOMEWORK 6 FOR 18.100B/C, SECTION 1, FALL 2010 DUE THURSDAY 28 OCTOBER BY 11AM IN CLASS OR 2-108 OR BY EMAIL BEFORE 5PM.

(1) Rudin Chap 4, No 2. If $f : X \longrightarrow Y$ is a continuous map between metric spaces, $E \subset X$ and $f(E) \subset Y$ is its image under f, show that the closures satisfy

$$f(\overline{E}) \subset \overline{f(E)}.$$

Give an example where the right side is strictly larger than the left.

(2) Consider the cartesian product $X \times Y = \{(x, y); x \in X,\ y \in Y\}$ of two metric spaces with the distance

$$D((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

If $f : X \longrightarrow Y$ is any map, define its graph by

$$G(f) = \{(x, y) \in X \times Y; x \in X,\ y = f(x)\}.$$

Show that if f is continuous then $G(f)$ is closed.

Show that if X is compact then a map $f : X \longrightarrow Y$ is continuous if and only if its graph is compact.

(3) Let X_1 and X_2 be closed subsets of a metric space X such that $X = X_1 \cup X_2$ and suppose $g_i : X_i \longrightarrow Y$, $i = 1, 2$, are two continuous maps defined on them. Show that if $g_1(x) = g_2(x)$ for all $x \in X_1 \cap X_2$ then $g : X \longrightarrow Y$ where $g(x) = g_i(x)$ for $x \in X_i$ is continuous.

(4) Let $\{y_n\}$ be a sequence in a metric space Y. Define a map on the set

$$D = \{1/n \in [0, 1]; n \in \mathbb{N}\} \longrightarrow Y$$

by $f(\frac{1}{n}) = y_n$. Show that f has a limit at 0 if and only if $\{y_n\}$ is convergent.

(5) Rudin Chap 4, No. 14. Show that any continuous map $f : [0, 1] \longrightarrow [0, 1]$ must have a fixed point, that is there exists at least one point $x \in [0, 1]$ such that $f(x) = x$.

Here are some questions on connected sets, designed to clarify things a little. They are for your amusement only.

(1) Recall that given a subset $E \subset X$ of a metric space we have defined the condition on a subset $F \subset E$ that it be relatively open (or relatively closed) and the characterization of this. Check that $F \subset E$ is relatively closed in E if and only if $F = \overline{F} \cap E$ where $\overline{F}$ is the closure in X. Show that a subset $E \subset X$ is connected if and only if the only decompositions of it into two disjoint relatively closed subsets $E = A \cup B$ has one of the sets empty.

(2) Suppose $f : X \longrightarrow Y$ is continuous and $E \subset X$, show that $f\big|_E : E \longrightarrow Y$ is continuous with the metric on E induced from X.

(3) Show that if and $f : X \longrightarrow Y$ is continuous and $U \subset f(X)$ is relatively open (resp. relatively closed) set then $f^{-1}(U)$ is open (resp. closed).

(4) Suppose $E \subset X$ is connected and $f : X \longrightarrow Y$ is continuous, show that if $f(E) = A \cup B$ is a decomposition into relatively closed subsets then

$E = (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) = f\big|_E^{-1}(A) \cup f\big|_E^{-1}(B)$ is a decomposition into relatively closed subsets.

(5) Deduce from this that the continuous image of a connected set is connected.