HOMEWORK 6 FOR 18.100B/C, SECTION 1, FALL 2010
DUE THURSDAY 28 OCTOBER

(1) Rudin Chap 4, No 2. If $f : X \longrightarrow Y$ is a continuous map between metric spaces, $E \subset X$ and $f(E) \subset Y$ is its image under f, show that the closures satisfy

$$f(\overline{E}) \subset \overline{f(E)}.$$

Give an example where the right side is strictly larger than the left.

Solution: Since $\overline{f(E)}$ is closed, $f^{-1}(\overline{f(E)})$ is closed by the assumed continuity of f. Moreover $E \subset f^{-1}(\overline{f(E)})$ since $E \subset f^{-1}(f(E))$. Since the closure of E is contained in any closed set containing E, $\overline{E} \subset f^{-1}(\overline{f(E)})$, but this implies that $f(x) \in \overline{f(E)}$ for each $x \in \overline{E}$, i.e. $f(\overline{E}) \subset \overline{f(E)}$.

As a counterexample to equality, consider $f : (0, \infty) \longrightarrow \mathbb{R}$ given by $f(x) = x$ with E taken as the domain, $E = (0, \infty)$ and hence closed. On the other hand the closure of the range is $[0, \infty)$ which is strictly larger than $f(E) = f(\overline{E})$.

(2) Consider the cartesian product $X \times Y = \{(x, y); x \in X,\ y \in Y\}$ of two metric spaces with the distance

$$D((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

If $f : X \longrightarrow Y$ is any map, define its graph by

$$G(f) = \{(x, y) \in X \times Y; x \in X,\ y = f(x)\}.$$

Show that if f is continuous then $G(f)$ is closed.

Show that if X is compact then a map $f : X \longrightarrow Y$ is continuous if and only if its graph is compact.

Solution: The distance between pairs (x, y) and (x', y') is the sum of the distances $d_X(x, x') + d_Y(y, y')$. It follows that a sequence (x_n, y_n) converges to a limit (x, y) if and only if both $x_n \to x$ and $y_n \to y$. Indeed, $(x_n, y_n) \in B((x, y), \epsilon)$ implies $x_n \in B(x, \epsilon)$ and $y_n \in B(y, \epsilon)$ and conversely $x_n \in B(x, \epsilon/2)$ and $y_n \in B(y, \epsilon/2)$ implies $(x_n, y_n) \in B((x, y), \epsilon)$. Thus all but a finite number of elements of the sequence $\{(x_n, y_n)\}$ lie in a given open ball around (x, y) if and only if the same is true for both $\{x_n\}$ and $\{y_n\}$. So, suppose $(x_n, y_n) \to (x, y)$ with $(x_n, y_n) \in G(f)$. This implies $y_n = f(x_n)$ by the definition of the graph and $x_n \to x$, $f(x_n) \to y$ by the observation above. However, f is assumed to be continuous, so $f(x_n) \to f(x)$ which implies that $y = f(x)$ and hence $(x, y) \in G(f)$. Thus every sequence in $G(f)$ which converges in $X \times Y$ has limit in $G(f)$ which is therefore closed.

Suppose first that X is compact and f is continuous. To see that $G(f)$ is compact, consider an infinite subset E of it. Since $(x, y) \in E$ implies $y = f(x)$ this can only be infinite if $E_1 = \{x \in X; (x, y) \in E\}$ is infinite. Since X is compact, this must have a limit point, which is therefore the limit of a sequence x_n converging in X with limit $x \in E_1$ and with all x_n distinct and not equal to x. The continuity of f shows that $f(x_n) \to f(x)$ and hence

$(x, f(x))$ is a limit point of E (since all the $(x_n, f(x_n))$ are distinct) so $G(f)$ is compact.

Conversely, suppose that $G(f)$ is compact (with the metric from $X \times Y$ of course). Consider a general $C \subset Y$ which is closed. Suppose $x_n \in f^{-1}(C)$ and $x_n \to x$ in X. Then $(x_n, f(x_n)) \in G(f)$ has a convergent subsequence $(x_{n_k}, f(x_{n_k})$ with limit $(x, y) \in G(f)$ (since it is closed) so $y = f(x) \in X$ and since $f(x_{n_k}) \to y$ in Y (as discussed above) $y = f(x) \in C$. Thus $x \in f^{-1}(C)$ which must therefore be closed and hence f must be continuous.

Here is a much more sophisticated proof of the second part based on the fact that the two 'projection' maps $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$ are both continuous. Namely for any map f the inverse image of a set $C \subset Y$ is precisely $f^{-1}(C) = \pi_1(X \times C \cap G(f))$. Indeed, if $x \in f^{-1}(C)$ then $y = f(x) \in C$ so there is a point $(x, f(x)) \in G(f) \cap (X \times C)$ with $\pi_1((x, f(x)) = x$. Conversely if $x = \pi_1(x, y)$ with $(x, y) \in G(f) \cap (X \times C)$ then $y = f(x) \in C$. Now, $X \times C = \pi_2^{-1}(C)$ so we have written $f^{-1}(C) = \pi_1(G(f) \cap \pi_2^{-1}(C))$. If C is closed and we assume that $G(f)$ is compact then $\pi_2^{-1}(C)$ is closed by the continuity of π_2, $G(f) \cap (\pi_2^{-1}(C))$ is compact by the assumed compactness of $G(f)$ and hence $f^{-1}(C) = \pi_1(G(f) \cap \pi_2^{-1}(C))$ is compact, by the continuity of π_1 hence closed. Thus f is continuous since the inverse image of any closed set is closed. The corresponding 'sophisticated' proof of the first part (proving $G(f)$) is compact) is to realize that if f is continous then the map $I_f : X \longrightarrow Y \times Y$ where $I_f(x) = (x, f(x))$ is also continuous – using for instance the convergence discussion above – so $G(f) = I_f(X)$ is compact if X is compact. You could also see the converse, assuming $G(f)$ is compact, by observing that $L_f : X \longrightarrow G(f)$, just I_f with the target space reduced to $G(f)$, is 1-1 and onto and has continuous inverse $\pi_1\big|_{G(f)}$ so, by a Theorem in Rudin, the inverse of π_1, i.e. $L_f : X \longrightarrow G(f)$ is continuous, but then $f = \pi_2 \cap L_f$ is also continuous.

So many possibilities it is hard to choose ....

(3) Let X_1 and X_2 be closed subsets of a metric space X such that $X = X_1 \cup X_2$ and suppose $g_i : X_i \longrightarrow Y$, $i = 1, 2$, are two continuous maps defined on them. Show that if $g_1(x) = g_2(x)$ for all $x \in X_1 \cap X_2$ then $g : X \longrightarrow Y$ where $g(x) = g_i(x)$ for $x \in X_i$ is continuous.

Solution: First note that $g : X \longrightarrow Y$ is a well-defined map, since if $x \in X$ then either $x \in X_1 \setminus X_2$ or $x \in X_2 \setminus X_2$ or $x \in X_1 \cap X_2$. In each case $g(x)$ is well defined as $g_1(x)$, $g_2(x)$ or the common value $g_1(x) = g_2(x)$.

Maybe the conceptually easiest way to see the continuity is to use sequences. Suppose $x_n \to x$ in X. Consider all those x_n which lie in X_1. If there are only finitely many then eventually $x_n \in X_2$ and hence $g(x_n) = g_2(x_n)$ for large n converges to $g_2(x) = g(x)$ since X_2 is closed (so $x \in X_2$) and g_2 is continuous. Otherwise there is a subsequence of x_n in X_1 which therefore converges to $x \in X_1$, since X_1 is closed, so along this subsequence $g_1(x_{n_k}) \to g_1(x) = g(x)$. Now, either the number of points not in X_1 is finite or else forms a subsequence in X_2, which therefore converges as before. In the first case it follows immediately $g(x_n) \to g(x)$ and also in the second, since give an open ball centered at $g(x)$ the image of the first subsequence under g_1 and the image of the complementary subsequence under g_2 both line in this ball from some integer onwards. Thus $g(x_n) \to g(x)$ in all cases and hence g is continuous.

It is probably easier to use closed sets. Suppose $C \subset Y$ is closed. Then $x \in g^{-1}(C)$ if either $x \in X_1$ and $g_1(x) \in C$ or $x \in X_2$ and $g_2(x) \in C$. Thus $g^{-1}(C) = g_1^{-1}(C) \cup g_2^{-1}(C)$. By assumption $g_1 : X_1 \longrightarrow Y$ and $g_2 : X_2 \longrightarrow Y$ are continuous, so $g_i^{-1}(C) \subset X_i$ for $i = 1, 2$, is closed *as a subset of X_i as a metric space.* However, since the X_i are closed, being relatively closed in X_i is the same as being closed in X. Thus $g^{-1}(C)$ is closed, as the union of two closed sets.

(4) Let $\{y_n\}$ be a sequence in a metric space Y. Define a map on the set
$$D = \{1/n \in [0, 1]; n \in \mathbb{N}\} \longrightarrow Y$$
by $f(\frac{1}{n}) = y_n$. Show that f has a limit at 0 if and only if $\{y_n\}$ is convergent.

Solution. This is just the definitions. Namely, f has a limit y at 0 if and only if, given $\epsilon > 0$ there exists $\delta > 0$ such that $0 < d(0, 1/n) = 1/n < \delta$ implies $d(y, f(1/n)) = d(y, y_n) < \epsilon$. Of course $y_n \to y$ if and only if given $\epsilon > 0$ there exists N such that $n > N$ implies $d(y, y_n) < \epsilon$. These are completely equivalent, just by taking $N = 1/\delta$.

(5) Rudin Chap 4, No. 14. Show that any continuous map $f : [0, 1] \longrightarrow [0, 1]$ must have a fixed point, that is there exists at least one point $x \in [0, 1]$ such that $f(x) = x$.

Solution: Consider the subsets $L = \{x \in [0, 1]; f(x) \leq x\}$ and $U = \{x \in [0, 1]; f(x) \geq x\}$. These are non-empty, since $0 \in L$ and $1 \in U$. They are both closed, since $g : [0, 1] \longrightarrow \mathbb{R}$ defined by $g(x) = f(x) - x$ is continuous and $L = g^{-1}((\infty, 0])$, $U = g^{-1}([0, \infty))$. A point $x \in [0, 1]$ satisfies $f(x) = x$ if and only if $x \in L \cap U$. The set $L \cap U$ cannot be empty, since if it was then L and U would be non-empty, disjoint, closed subsets with $L \cup U = [0, 1]$ contradicting the known connectedness of $[0, 1]$. Thus at least one such fixed point must exist.

Here are some questions on connected sets, designed to clarify things a little. They are for your amusement only.

(1) Recall that given a subset $E \subset X$ of a metric space we have defined the condition on a subset $F \subset E$ that it be relatively open (or relatively closed) and the characterization of this. Check that $F \subset E$ is relatively closed in E if and only if $F = \overline{F} \cap E$ where $\overline{F}$ is the closure in X. Show that a subset $E \subset X$ is connected if and only if the only decompositions of it into two disjoint relatively closed subsets $E = A \cup B$ has one of the sets empty.
(2) Suppose $f : X \longrightarrow Y$ is continuous and $E \subset X$, show that $f\big|_E : E \longrightarrow Y$ is continuous with the metric on E induced from X.
(3) Show that if and $f : X \longrightarrow Y$ is continuous and $U \subset f(X)$ is relatively open (resp. relatively closed) set then $f^{-1}(U)$ is open (resp. closed).
(4) Suppose $E \subset X$ is connected and $f : X \longrightarrow Y$ is continuous, show that if $f(E) = A \cup B$ is a decomposition into relatively closed subsets then $E = (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) = f\big|_E^{-1}(A) \cup f\big|_E^{-1}(B)$ is a decomposition into relatively closed subsets.
(5) Deduce from this that the continuous image of a connected set is connected.