HOMEWORK 5 FOR 18.100B AND 18.100C, FALL 2010

Paper solutions Due 11AM, Thursday, October 21, in lecture or 2-108, Electronic submission to rbm at math dot mit dot edu by 5PM.

HW5.1 Modified version of Rudin Ch 3 No 1. Prove that for a sequence s_n in $\mathbb{C}$, the convergence of s_n implies the convergence of $|s_n|$. Give a counterexample to the converse statement.

HW5.2 Rudin Ch 3 No 7. Show that if $a_n \geq 0$ and $\sum\limits_n a_n$ converges then $\sum\limits_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

HW5.3 Rudin Ch 3 No 16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define a sequence x_2, x_3,... by the recursion formula

$$x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}).$$

(a) Prove that x_n decreases monotonically and that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

(b) Set $\epsilon_n = x_n - \sqrt{\alpha}$ and and show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

so that if $\beta = 2\sqrt{\alpha}$ then

$$\epsilon_{n+1} < \beta(\frac{\epsilon_1}{\beta})^{2^n}.$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\epsilon_1/\beta < 1/10$ and therefore

$$\epsilon_5 < 4 \cdot 10^{-16},\ \epsilon_6 < 4 \cdot 10^{-32}.$$

HW5.4 Let $f : X \longrightarrow Y$ be a map between sets. Let $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ be the *power sets* – the collection of subsets respectively of X and Y. Define maps

$$f_\# : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y),\ f_\#(A) = \{y \in Y; \exists\ a \in A \text{ with } y = f(a)\}$$
$$f^\# : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X),\ f^\#(B) = \{x \in X; f(x) \in B\}$$

(usually denoted as f and f^{-1} respectively). Compute the two composite maps $f^\# \circ f_\# : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ and $f_\# \circ f^\# : \mathcal{P}(Y) \longrightarrow \mathcal{P}(Y)$.

HW5.5 Show that for each fixed point p in a metric space X the distance from p, $f(x) = d(x,p)$ defines a continuous function $f : X \longrightarrow \mathbb{R}$.