HOMEWORK 5 FOR 18.100B AND 18.100C, FALL 2010 SOLUTIONS

Paper solutions Due 11AM, Thursday, October 21, in lecture or 2-108, Electronic submission to rbm at math dot mit dot edu by 5PM.

HW5.1 Modified version of Rudin Ch 3 No 1. Prove that for a sequence s_n in $\mathbb{C}$, the convergence of s_n implies the convergence of $|s_n|$. Give a counterexample to the converse statement.

Solution: If $s_n \to s$ for a sequence in $\mathbb{C}$ then given $\epsilon > 0$ there exists N such that for all $n > N$, $|s_n - s| < \epsilon$. By the 'reverse triangle inequality' $||s_n| - |s|| \le |s_n - s|$ so $|s_n| \to |s|$. A counterexample to the converse is $s_n = (-1)^n$ which does not converge, but of course $|(-1)^n| = 1$ does converge.

HW5.2 Rudin Ch 3 No 7. Show that if $a_n \ge 0$ and $\sum\limits_n a_n$ converges then $\sum\limits_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

Solution. The Cauchy Schwartz inequality (for any finite N) shows that

$$(\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n})^2 \le \sum_{n=1}^{N} a_n \cdot \sum_{n=1}^{N} \frac{1}{n^2}.$$

since both terms on the right are bounded independent of N, since both series converge, the sequence of partial sums on the left is bounded, hence the series itself (having positive terms) converges.

HW5.3 Rudin Ch 3 No 16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define a sequence x_2, x_3,... by the recursion formula

$$x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}).$$

(a) Prove that x_n decreases monotonically and that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

(b) Set $\epsilon_n = x_n - \sqrt{\alpha}$ and and show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

so that if $\beta = 2\sqrt{\alpha}$ then

$$\epsilon_{n+1} < \beta(\frac{\epsilon_1}{\beta})^{2^n}.$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\epsilon_1/\beta < 1/10$ and therefore

$$\epsilon_5 < 4 \cdot 10^{-16}, \ \epsilon_6 < 4 \cdot 10^{-32}.$$

Solution:

(a) We can compute $x_{n+1} - x_n = \frac{1}{2}(\frac{\alpha}{x_n} - x_n)$. Thus, if $x_n > \sqrt{\alpha}$ then $x_{n+1} - x_n < 0$ and $x_{n+1}^2 - \alpha = \frac{1}{4}(x_n^2 - 2\alpha + \frac{\alpha^2}{x_n^2}) > 0$. Thus $x_n > \sqrt{\alpha}$ and $x_{n+1} < x_n$ are true by induction.

Since the sequence decreases monotonically and is bounded below by $\sqrt{\alpha}$, it converges to a limit $t = \sqrt{\alpha} + r$ for some $r \geq 0$. Since $\alpha/x_n \to \alpha/t$ and $x_{n+1} \to t$ the limit must satisfy $t = \frac{1}{2}(t + \frac{\alpha}{t})$ and hence $t^2 = \alpha$ so $t = \sqrt{\alpha}$.

(b) If $\epsilon_n = x_n - \sqrt{\alpha}$ then $\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2}(x_n - 2\sqrt{\alpha} + \frac{\alpha}{x_n}) = (x_n - \sqrt{\alpha})^2/2x_n = \epsilon_n^2/2x_n < \epsilon_n^2/2\sqrt{\alpha}$ since $x_n > \sqrt{\alpha}$ for all n. Setting $\beta = 2\sqrt{\alpha}$ this becomes $\frac{\epsilon_{n+1}}{\beta} < \left(\frac{\epsilon_n}{\beta}\right)^2$ for all $n > 0$ and so by induction $\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2n}$.

(c) You can do the computation!

HW5.4 Let $f : X \longrightarrow Y$ be a map between sets. Let $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ be the *power sets* – the collection of subsets respectively of X and Y. Define maps

$$f_\# : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y),\ f_\#(A) = \{y \in Y; \exists\ a \in A \text{ with } y = f(a)\}$$
$$f^\# : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X),\ f^\#(B) = \{x \in X; f(x) \in B\}$$

(usually denoted as f and f^{-1} respectively). Compute the two composite maps $f^\# \circ f_\# : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ and $f_\# \circ f^\# : \mathcal{P}(Y) \longrightarrow \mathcal{P}(Y)$.

Remark. I also meant to ask you to show that $f^\# \circ f_\#(A) \supset A$ where as $f_\# \circ f^\#(B) \subset B$ for all subsets $A \subset X$ and $B \subset Y$.

Solution: This question was maybe a bit open-ended and was really intended to get you thinking about these maps on sets, since they are used in discussing continuity. I had in mind something like:

$$\begin{aligned} f^\# \circ f_\#(A) &= \{x \in X; \exists\ x' \in A \text{ with } f(x) = f(x')\} \\ f_\# \circ f^\#(B) &= f(X) \cap B\ \forall\ B \subset Y. \end{aligned} \tag{1}$$

Both these statements follow directly. Since $x \in f^\# \circ f_\#(A)$ means precisely that $f(x) \in f_\#(A)$ which is exactly the condition that there exists $x' \in A$ such that $f(x) = f(x')$. Similarly, $y \in f_\# \circ f^\#(B)$ means just that there exists $x \in f^\#(B)$ with $f(x) = y$ and hence that $y \in B \cap f(X)$ and conversely.

The part I did not ask follows directly from (1).

HW5.5 Show that for each fixed point p in a metric space X the distance from p, $f(x) = d(x, p)$ defines a continuous function $f : X \longrightarrow \mathbb{R}$.

Solution: By the triangle (reverse) inequality,

$$|f(x) - f(x')| = |d(x,p) - d(x',p)| \leq d(x, x')$$

so given $\epsilon > 0$ choosing $\delta = \epsilon$, $d(x, x') < \delta$ implies that $d_{\mathbb{R}}(f(x), f(x')) < \epsilon$, so f is continuous.