HOMEWORK 4 FOR 18.100B AND 18.100C, FALL 2010

Paper solutions Due 11AM, Thursday, October 7 in lecture or 2-108, Electronic submission to rbm at math dot mit dot edu by 5PM.

As usual, clarity is especially prized.

HW4.1 Rudin Chap 2, 22. A metric space is said to be *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

HW4.2 Rudin Chap 2, 23 (reworded). Prove that for every separable metric space there is a countable collection $\{B_j\}_{j\in\mathbb{N}}$, of open balls (neighborhoods to Rudin) with the property that for any open set G and any $x \in G$ there is a B_j such that $x \in B_j \subset G$.

HW4.3 Rudin Chap 2, 24. Prove that any metric space with the property that every infinite subset has a limit point is separable. Hint – show that for each $n \in \mathbb{N}$ there are finitely many balls or radius $1/n$ which together cover the metric space (otherwise there is an infinite set with all points distant at least $1/n$ apart).

HW4.4 Rudin Chap 2, 26. Let X be a metric space in which every infinite subset contains a limit point, prove that X is compact. Hint – Combining the preceding two questions conclude that there is a collection of balls $\{B_j\}$ as above and use this to show that every open cover of X has a countable subcover. Thus it suffices to show that every countable open cover G_j has a finite subcover. If not, show that the closed sets $F_n = X \setminus \bigcup_{k=1}^{n} G_k$ decrease as n increases and are infinite but that $\bigcap_n F_n = \emptyset$. So we can choose a countably infinite set E with the nth point in F_n. However a limit point of this set would be in each F_n, so ....

HW4.5 Rudin Chap 2, 29. Prove that any open set in $\mathbb{R}$ is the union of a collection of disjoint open intervals which is at most countable.