HOMEWORK 4 FOR 18.100B AND 18.100C, FALL 2010 SOLUTIONS, SOMEWHAT WORDY.

As usual the problems will each be worth 10 points and clarity is especially prized.

HW4.1 Rudin Chap 2, 22:- A metric space is said to be *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

Solution. The subset $\mathbb{Q}^k$, consisting of the k-tuples of real numbers all of whose entries are rational, is dense. To see this, recall that between any two distinct real numbers there is a rational. So if $\epsilon > 0$ is given and $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ then there exists, for each $j = 1, \dots, k$ some $q_j \in \mathbb{Q}$ satisfying $x_j < q_k < \epsilon/k$. It follows that if $q = (q_1, \dots, q_k)$ then

$$|x - q| = (\sum_{j=1}^{k} |x_j - q_j|^2)^{\frac{1}{2}} < k^{-\frac{1}{2}}\epsilon \le \epsilon.$$

Thus each ball with positive radius around any point $x \in \mathbb{R}^k$ contains a point of $\mathbb{Q}^k$, which is therefore dense in $\mathbb{R}^k$.

We also know that $\mathbb{Q}$ is countable, hence $\mathbb{Q}^k$ is countable as a (finite) product of countable sets. Thus $\mathbb{R}^k$ is separable since it has a countable dense subset.

HW4.2 Rudin Chap 2, 23 (reworded):- Prove that for every separable metric space there is a countable collection $\{B_j\}_{j\in\mathbb{N}}$, of open balls (neighborhoods to Rudin) with the property that for any open set G and any $x \in G$ there is a B_j such that $x \in B_j \subset G$.

Solution: Let X be the metric space. By assumption it is separable, so let D be a countable dense subset. Then consider the collection, $\mathcal{B}$ of all open balls with centers from D and radius $1/n$ for some $n \in \mathbb{N}$. This is a countable collection since it is in 1-1 correspondence with $D \times \mathbb{N}$ which is countable as the product of (at most in the case of D) countable sets.

Now, suppose $G \subset X$ is open and $x \in G$. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset G$. Since D is dense in X, given $n \in \mathbb{N}$ with $n > 2/\epsilon$, there exists some $p \in D$ such that $x \in B(p, 1/n) \in \mathcal{B}$. By the triangle inequality, if $y \in B(p, 1/n)$ then $d(y, x) \le d(y, p) + d(p, x) \le 2/n < \epsilon$ so $y \in B(x, \epsilon) \subset G$. Thus $x \in B(p, 1/n) \subset G$ as desired and $\mathcal{B}$ satisfies the required property.

HW4.3 Rudin Chap 2, 24:- Prove that any metric space with the property that every infinite subset has a limit point is separable. Hint – show that for each $n \in \mathbb{N}$ there are finitely many balls or radius $1/n$ which together cover the metric space (otherwise there is an infinite set with all points distant at least $1/n$ apart).

Solution: Let X be a metric space with the property that every infinite subset of it has a limit point. For each $n \in \mathbb{N}$ choose a subset of X by first choosing one point. Then, if possible, choose a second point at least distance $1/n$ from the first. Proceed in this way, at each stage choosing a point distant $1/n$ or more from all the previous choices. At some point no further choices are possible since a set with all points distant at least $1/n$ from each other cannot have a limit point, so an infinite number of such choices in X is not possible. Thus, for each n this procedure gives a finite set such that the balls of radius $1/n$ with elements of this set as centers covers X. The union of these finite sets is a(n at most) countable set which is dense in X since for every $x \in X$ and every $\epsilon > 0$ one can choose $n > 1/\epsilon$ and then there is a point in the set distant at most $1/n < \epsilon$ from x.

HW4.4 Rudin Chap 2, 26. Let X be a metric space in which every infinite subset contains a limit point, prove that X is compact. Hint – Combining the preceding two questions conclude that there is a collection of balls $\{B_j\}$ as above and use this to show that every open cover of X has a countable subcover. Thus it suffices to show that every countable open cover G_j has a finite subcover. If not, show that the closed sets $F_n = X \setminus \bigcup_{k=1}^n G_k$ decrease as n increases and are infinite but that $\bigcap_n F_n = \emptyset$. So we can choose a countably infinite set E with the nth point in F_n. However a limit point of this set would be in each F_n, so ....

Solution: If X is a metric space in which every infinite set has a limit point, we know that X is separable by the preceding problem. Then choose a countable collection of balls $\mathcal{B}$ as HW4.2. Now, given an open cover G_α of X for each $x \in X$ there is a $B \in \mathcal{B}$ such that $x \in B \subset G_\alpha$ for some α. For each $B \in \mathcal{B}$ either choose an α such that $B \subset G_\alpha$, if there is one, or else do nothing. This determines an at most countable subcover of the G_α since every $x \in X$ is contained in one of the balls which *are* contained in a G_α.

Thus, we can consider a countable subcover G_j – if it is finite we are already done. The sets

$$F_n = X \setminus (\bigcup_{j=1}^n G_j)$$

are closed, as the complements of open sets, and decrease with n. To say that the G_j cover is to say $\bigcap_n F_n = \emptyset$. Suppose that no F_n is empty. Then we can choose a subset E of X by choosing successive elements $x_n \in F_n$. This set must be infinite, since if it were finite we must have made the same choice infinitely often, and since the F_n are getting smaller this would mean there was a point in $\bigcap_n F_n$. Thus, E must have a limit point. Now, for each n all but a finite number of points of E are in F_n thus the limit point must also be a limit point of $E \cap F_n$ for each n. Hence, since they are closed, it must be in F_n for each n. Thus, again, we have found a point in $\bigcap_n F_n$ so the assumption that each F_n is non-empty must be false. Hence $X = \bigcup_{j=1}^n G_j$ for some n so G_j does indeed have a finite subcover. Thus X is compact since it has the property that every open cover has a finite subcover.

HW4.5 Rudin Chap 2, 29. Prove that any open set in $\mathbb{R}$ is the union of a collection of pairwise disjoint open intervals which is at most countable. Note – the pairwise was added afterwards, since a few people were confused by the meaning of 'disjoint' otherwise.

Solution: Observe that the union of any collection of open intervals which all contain a common point is an open interval (possibly infinite) with end points the infimum of the lower end points (if this set is not bounded below then $-\infty$) or supremum of the upper end points (or $+\infty$ if this set is not bounded above). Now, take a point in $O \cap \mathbb{Q}$ and consider the union of all the open intervals which contain it and are contained in O. Then, if possible, select a point in $O \cap \mathbb{Q}$ which is not in this first interval and proceed. This constructs an at most countable collection of intervals which are contained in O and together cover it. They must be disjoint since if two have non-empty intersection the union is an interval which would contain any point in either, so must be equal to the first.