HOMEWORK 3 FOR 18.100B AND 18.100C, FALL 2010
DUE THURSDAY, SEPTEMBER 30 IN 2-108.

As usual, physical homework due in 2-108 by 11AM. Electronic submission (to rbm at math dot mit dot edu) up to 5PM.

HW3.1 Let X be a set with the discrete metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

Which subsets of X are compact? Of course you should justify your answer.

HW3.2 Rudin, Chap. 2, Problem 9 extended a little: Let E° denote the set of all interior points of a set E (called the *interior of* E) in a metric space X – recall that an interior point of E is a point $p \in E$ such that $B(p,\epsilon) \subset E$ for some $\epsilon > 0$.

(a) Prove that E° is open.
(b) Prove that E is open if and only if $E^\circ = E$.
(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
(d) Prove that the complement of E° is the closure of the complement of E.
(e) Show that E° is the union of all open sets contained in E.
(f) Do E and $\overline{E}$ always have the same interiors?
(g) Do E and E° necessarily have the same closures?

HW3.3 Rudin Chap. 2, Problem 12: Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

HW3.4 Rudin, Chap. 2, Problem 16: Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p,q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

HW3.5 Prove that every compact metric space has a countable dense subset. Hint: For each natural number n look at the open cover given by all open balls of radius 1/n, use compactness to get a finite subcover and look at all the centers of the balls in these finite subcovers.