HOMEWORK 3 FOR 18.100B/C, FALL 2010
SOLUTIONS

As usual, physical homework due in 2-108 by 11AM. Electronic submission (to rbm at math dot mit dot edu) up to 5PM.

HW3.1 Let X be a set with the discrete metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

Which subsets of X are compact? Of course you should justify your answer.

Solution: The only subsets of X with the discrete metric which are compact are finite sets. Certainly finite sets are compact in any metric space, since any open cover has a finite subcover, given by choosing an element of the cover which contains each remaining point in the set successively. Conversely if $K \subset X$ then consider the open cover consisting of the open balls $B(x, \frac{1}{2})$ of radius $\frac{1}{2}$ for each $x \in K$. The ball $B(x, \frac{1}{2})$ contains only the point x of X so if this cover has a finite subcover, which it must if K is compact, then K is finite since this open cover presents it as a finite union of points.

HW3.2 Rudin, Chap. 2, Problem 9 extended a little: Let E° denote the set of all interior points of a set E (called the *interior of* E) in a metric space X – recall that an interior point of E is a point $p \in E$ such that $B(p, \epsilon) \subset E$ for some $\epsilon > 0$.

(a) Prove that E° is open.
(b) Prove that E is open if and only if $E^\circ = E$.
(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
(d) Prove that the complement of E° is the closure of the complement of E.
(e) Show that E° is the union of all open sets contained in E.
(f) Do E and $\overline{E}$ always have the same interiors?
(g) Do E and E° necessarily have the same closures?

Solution:

(a) By definition, if $x \in E^\circ$ then for some $\epsilon > 0$, $B(x, \epsilon) \subset E$. Since $B(x, \epsilon)$ is open, for each $y \in B(x, \epsilon)$, there exists $\delta > 0$, in fact it suffices to take $\delta < \epsilon - d(x, y)$, such that $B(y, \delta) \subset B(x, \epsilon) \subset E$. Thus in fact $B(x, \epsilon) \subset E^\circ$ which is therefore open.
(b) Certainly if $E = E^\circ$ then E is open by the preceding result. If E is open then for each $x \in E$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset E$, so $x \in E^\circ$ and hence $E^\circ = E$.
(c) If $G \subset E$ is open then for each $x \in G$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset G \subset E$ so $x \in E^\circ$ and hence $G \subset E^\circ$.
(d) Since E° is open, $X \setminus E^\circ$ is closed, but $E^\circ \subset E$ implies $(X \setminus E) \subset (X \setminus E^\circ)$ so $X \setminus E^\circ$ is a closed set containing $X \setminus E$ which implies, by a result in class/Rudin, that $\overline{X \setminus E} \subset X \setminus E^\circ$. Conversely, if $X \setminus E \subset C$

and C is closed, then $G = (X \setminus C) \subset E$ is open, so by the result above, $G \subset E^\circ$ which implies $C \supset (X \setminus E^\circ)$. Thus

$$X \setminus E^\circ = \bigcap\{C; C \supset (X \setminus E),\ C \text{ closed}\} \Longrightarrow X \setminus E^\circ = \overline{X \setminus E}$$

again by a result in class/Rudin.

(e) As shown above, E° contains all open (in X) subsets of E and is itself open, so $E^\circ = \bigcup\{G \subset E; G \text{ open}\}$.

(f) No, not in general. For instance $\mathbb{Q} \subset \mathbb{R}$ has empy interior whereas its closure, $\mathbb{R}$ is open.

(g) No, not in general. For instance if E consists of one point in $\mathbb{R}$ then it has no interior so the closure of its interior is strictly smaller than E.

HW3.3 Rudin Chap. 2, Problem 12: Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Solution: Consider any open cover of K, $K \subset \bigcup_{a\in A} O_a$. There must be at least one $a_1 \in A$ such that $0 \in O_{a_1}$, since $0 \in K$. Since this O_{a_1} is open, there exists $\epsilon > 0$ such that $B(0, \epsilon) \subset O_{a_1}$. By the Archimedean principle, there exists N such that if $n > N$ then $1/n < \epsilon$ which implies $1/n \in O_{a_1}$. Thus, all but a finite number of points, namely $\{1/n; n \le N\}$ of K lie in this O_{a_1} and each of these lies in (at least) one of the open sets, so the open cover has a finite subcover.

HW3.4 Rudin, Chap. 2, Problem 16: Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

Solution: Now we know about the reals, and $\mathbb{Q} \subset \mathbb{R}$ has the same metric, so we can use the properties of relatively open and closed sets. We know that $\sqrt{2}$ and $\sqrt{3}$ are not rational, hence

$$E = \{p \in \mathbb{Q}; \sqrt{2} \le p \le \sqrt{3}\} = [\sqrt{2}, \sqrt{3}] \cap \mathbb{Q} \tag{1}$$

is closed in $\mathbb{Q}$ by the properties of relatively closed sets. [Or you can prove it directly of course.] This also shows that E is bounded.

The open sets (in $\mathbb{Q}$ mind!) $O_n = (\sqrt{2}, \sqrt{3} - 1/n/) \cap \mathbb{Q}$ cover E since if $p \in E$ then $\sqrt{3} - p > 0$ so $n \in \mathbb{N}$ large enough, $\sqrt{3} - p > 1/n$ implies $p \in O_n$. These sets increase as n increases, so if there was a finite subcover then $E \subset O_n$ for some n but by the density of the rationals in the reals, there exists $p \in \mathbb{Q}$ such that $\sqrt{3} - 1/n < p < \sqrt{3}$ and for $n > 1$ large this implies that $p \in E$ but $p \notin Q_n$. Thus E cannot be compact since this open cover has no finite subcover.

You could also construct an infinite subset without a limit point in E.

It does follow that E is open by the same type of argument as above, namely

$$E = \{p \in \mathbb{Q}; \sqrt{2} < p < \sqrt{3}\} = (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q} \tag{2}$$

is open in $\mathbb{Q}$ by the properties of relatively open subsets.

HW3.5 Prove that every compact metric space has a countable dense subset. Hint: For each natural number n look at the open cover given by all open balls of radius $1/n$, use compactness to get a finite subcover and look at all the centers of the balls in these finite subcovers.

Solution: Just as the hint says! Let X be a compact metric space and consider the open cover, for $n \in \mathbb{N}$ fixed,

$$X = \bigcup_{x \in X} B(x, 1/n). \tag{3}$$

By the assumed compactness this must have a finite subcover, let D_n be the set of centers of some such finite subcover. Doing this for each n let $D = \bigcup_n D_n$ be the union, which is (at most) countable. Thus, if $x \in X$ and $\epsilon > 0$ then there exists $n \in \mathbb{N}$ such that $1/n < \epsilon$. By the covering property, there exists $y \in D_n \subset D$ such that $x \in B(y, 1/n)$ which implies $y \in B(x, 1/n) \subset B(x, \epsilon)$. Thus every open ball around an arbitrary point, $x \in X$, contains a point of D. Thus, D is dense in X which therefore has a countable dense subset.