HOMEWORK 2 FOR 18.100B/C SECTION 1, FALL 2010
SOLUTIONS AND REMARKS

HW2.1 Rudin Chap 1, Prob 13: If x and y are complex numbers show that

$$||x| - |y|| \le |x - y|.$$

Solution: Taking the squares of both sides this is equivalent to

$$|x|^2 - 2|x||y| + |y|^2 \le (x - y)(\overline{x} - \overline{y}) = |x|^2 - x\overline{y} - y\overline{x} + |y|^2$$

where we use the definition that $|x|^2 = a^2 + b^2 = (a + ib)(a - ib) = x\overline{x}$ for a complex number $x = a + ib$ where $\overline{x} = a - ib$. Thus, dropping the square from both sides and setting $y = c + id$ this is the same as

$$x\overline{y}+y\overline{x} = (a+ib)(c-id)+(a-ib)(c+id) = 2ac+2bd \le 2|x||y| = 2(a^2+b^2)^{\frac{1}{2}}(c^2+d^2)^{\frac{1}{2}}.$$

This we know is Cauchy's inequality in $\mathbb{R}^2$ from class.

Note, it is not really legal to use exponential functions, sines or cosines since while what you know about these is (I hope) correct, we have not proved these facts and there would be great danger of circular arguments if you used them to prove elementary things like this.

HW2.2 Let (X, d) be a metric space – i.e. d is a metric on X. Define the (British Railway) 'metric' on X by choosing a point $L \in X$ and defining

(a) $d_{\mathrm{BR}}(x, x) = 0, \ \forall\, x \in X.$

(b) $d_{\mathrm{BR}}(x, y) = d(x, L) + d(L, y)$ if $x \ne y$.

Show that this is indeed a metric and that every subset of X not containing L is open. (Note that L represents London!)

Solution: Symmetry is immediate from the definition and $d_{\mathrm{BR}}(x, y) > 0$ if $x \ne y$ since then either $x \ne L$ or $y \ne L$. Similarly the triangle inequality

$$d_{\mathrm{BR}}(x, y) \le d_{\mathrm{BR}}(x, z) + d_{\mathrm{BR}}(z, y)$$

is trivially true if $x = y$ and otherwise becomes

$$d(x, L) + d(L, y) \le d(x, L) + d(L, z) + d(z, L) + d(L, y), \text{ if } z \ne x \text{ and } z \ne y,$$

which follows from the triangle inequality for d. The remaining case reduces to $x = z$ and $z \ne y$ by symmetry when the desired inequality is just

$$d(x, L) + d(L, y) \le d(z, L) + d(L, y)$$

which is then trivial.

Consider a subset $E \subset X$ which does not contain L. Then for each point $x \in E$, $d(x, L) > 0$. Consider the ball for the new metric

$$B_{\mathrm{BR}}(x, \frac{1}{2}d(x, L)) = \{y \in X; d_{\mathrm{BR}}(x, y) < \frac{1}{2}d(x, L)\} = \{x\}$$

since if $y \ne x$ then $d_{\mathrm{BR}}(x, y) = d(x, L) + d(y, L) > \frac{1}{2}d(x, L)$. Thus this open ball is contained in E which is therefore open.

HW2.3 Rudin Chap 2, Prob 4. Is the set of irrational numbers countable? (Justify your answer!)

Solution: The real numbers are 'known' not to be countable. Let $\mathbb{I}$ be the set of irrational numbers. Then $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$ is the union of the rational and the irrational numbers. Now, $\mathbb{Q}$ is countable, so if $\mathbb{I}$ was also countable, $\mathbb{R}$ would also be countable since a countable union of countable sets is countable. Thus the irrational numbers must be uncountable.

HW2.4 Rudin Chap 2, Prob 5. Construct a bounded subset of $\mathbb{R}$ which has exactly three limit points.

Solution: There are plenty of possibilities! A subset of $\mathbb{R}$ with just one limit point is $E_0 = \{1/n,\ n \in \mathbb{N}\}$. Namely, 0 is a limit point since for any $\epsilon >$ there exists $n \in \mathbb{N}$ such that $n\epsilon > 0$ which implies that $1/n \in B(0, \epsilon)$ so $B(0, \epsilon) \cap E_0 \neq \emptyset$ for all $\epsilon > 0$. Conversely, there is no other limit point since if $x \in \mathbb{R}$, $x \neq 0$, the ball $B(x, \frac{1}{2}|x|)$ meets E_0 in a subset of $\{n; n|x| < 2\}$ which is finite (since $n|x| > 2$ implies $1/n < |x|/2$ so $|x - 1/n| < |x|/2$). Now, let $E_1 = \{1 + 1/n,\ n \in \mathbb{N}\}$, $E_2 = \{2 + 1/n,\ n \in \mathbb{N}\}$ and set $E = E_0 \cup E_1 \cup E_2$. The argument above shows that E_1 and E_2 each have one limit point, respectively 1 and 2 so these are also limit points of E. A sufficiently small ball around any other point $x \neq 0$, 1 or 2, meets each of E_0, E_1 and E_2 in a finite set, so also meets E in a finite set and hence is not a limit point.

HW2.5 In each case determine whether the given function d is a metric (Prove your answer!).

(a) Fix $N \in \mathbb{N}$. Let X be the set of sequences of zeroes and ones of total length N. For two sequences x, y let $d(x, y)$ be the number of places at which the two sequences differ.
(b) $X = \mathbb{R}$, $d(x, y) = (x - y)^2$.
(c) $X = \mathbb{R}$, $d(x, y) = \sqrt{|x - y|}$.
(d) $X = \mathbb{R}$, $d(x, y) = |x - 2y|$.
(e) $X = \mathbb{R}$, $d(x, y) = \frac{|x-y|}{1+|x-y|}$.

Solution:

(a) First consider the case $N = 1$. Thus our set $X_1 = \{0, 1\}$ with the discrete metric which we can denote $d_1(x, y)$. For general N we can write the putative distance function as

$$d(x, y) = \sum_j d_1(x_j, y_j)$$

since the right side is precisely the number of places at which the sequences differ. Now it is easy to check the three conditions. Surely $d(x, y)$ is well-defined, non-negative and vanishes if and only if the two finite sequences are equal. Symmetry is immediate, from symmetry for d_1 and the triangle inequality follows from the triangle inequality for d_1 by summing. Thus d is indeed a metric.

(b) Consider the three points 4, 2 and 0. By definition

$$d(4, 0) = 4^2 = 16,\ d(2, 0) + d(2, 4) = 2^2 + 2^2 = 8$$

so the triangle inequality is not always true and this is not a metric.

(c) Symmetry and the first axiom are clearly true. Expand out the square of what should be the right side of the triangle inequality

$$(\sqrt{|x-z|}+\sqrt{|z-y|})^2 \geq |x-z|+|z-y| \geq |x-y| = (\sqrt{|x-y|})^2.$$

For non-negative real numbers $a^2 \geq b^2$ is equivalent to $a \geq b$ so this proves the triangle inequality and $\sqrt{|x-y|}$ is indeed a metric.

(d) This is not a metric, since for instance $d(1,1)=1$ violates the first condition.

(e) This is a metric. Certainly $d(x,y) \geq 0$ is well-defined and vanishes only when $|x-y|=0$, so $x=y$. Symmetry is also immediate. For the triangle inequality we write out the desired right side

$$d(x,z)+d(z,y) = \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} = \frac{|x-y|+|y-z|+2|x-y||y-z|}{1+|x-y|+|y-z|+|x-y||y-z|}$$

So, for this to be $\geq \frac{|x-y|}{1+|x-y|}$ is equivalent (multiplying through by the positive denominators) to

(1)
$$|x-y|+|x-y||x-z|+|x-y||y-z|+|x-y||x-z||y-z| = |x-y|(1+|x-z|+|y-z|+|x-z||y-z|) \leq (|x-z|+|y-z|+2$$

We can drop the second term on the very right and the final factor of 2 to see that this is indeed true.

Note that this last case is actually used significantly in the theory of Fréchet spaces – which are special sorst of metric spaces which you might meet at some future date. Fréchet was in fact the 'inventor' (one would usually say 'discoverer', or 'formalized the notion') of metric spaces.